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IOANNIS BOLYAI DE BOLYA

APPENDIX

SCIENTIAM SPATII ABSOLUTE VERAM EXHIBENS.



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# IOANNIS BOLYAI DE BOLYA APPENDIX

SCIENTIAM SPATII ABSOLUTE VERAM EXHIBENS: A VERITATE AUT FALSITATE  
AXIOMATIS XI. EUCLIDEI, A PRIORI HAUD UNQUAM DECIDENDA, INDEPENDENTEM:  
ADIECTA AD CASUM FALSITATIS QUADRATURA CIRCULI GEOMETRICA.

EDITIO NOVA

OBLATA AB ACADEMIA SCIENTIARUM HUNGARICA  
AD DIEM NATALEM CENTESIMUM AUCTORIS CONCELEBRANDUM.

EDIDERUNT

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ACADEMIÆ SCIENTIARUM HUNGARICÆ SODALES.



LIPSIAE,  
IN ÆDIBUS B. G. TEUBNERI.

\*

[BUDAPESTINI, SUMPTIBUS ACADEMIÆ SCIENTIARUM HUNGARICÆ.]

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## **A P P E N D I X.**

**SCIENTIAM SPATII *absolute veram* exhibens :  
*a veritate aut falsitate Axiomatis XI Euclidei*  
*(a priori haud unquam decidenda) in-*  
*dependentem; adjecta ad casum fal-*  
*sitatis, quadratura circuli*  
*geometrica.***

---

**Auctore JOHANNÉ BOLYAI de eadem, Geometrarum  
in Exercitu Caesareo Regio Austriaco Ca-  
strensium Capitaneo.**

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## EXPLICATIO SIGNORUM.

- $\overline{ab}$  denotet complexum *omnium* punctorum cum punctis  $a, b$  in recta sitorum.
- $\overline{ab}$  • rectæ  $\overline{ab}$  in  $a$  bifariam sectæ dimidium illud, quod punctum  $b$  complectitur.
- $\overline{abc}$  • complexum *omnium* punctorum, quæ cum punctis  $a, b, c$  (non in eadem recta sitis) in eodem plano sunt.
- $ab\tilde{c}$  • plani  $\overline{abc}$  per  $\overline{ab}$  bifariam secti dimidium, punctum  $c$  complectens.
- $abc$  • portionum, in quas  $\overline{abc}$  per complexum rectarum  $b\tilde{a}, b\tilde{c}$  dividitur, *minorem*; sive *angulum*, cuius  $b\tilde{a}, b\tilde{c}$  crura sunt.
- $abcd$  • (si  $d$  in  $abc$  sit et  $b\tilde{a}, c\tilde{d}$  se invicem non secant) portionem ipsius  $abc$  inter  $b\tilde{a}, bc, c\tilde{d}$  comprehensam;  $bac\tilde{d}$  vero portionem plani  $\overline{abc}$  inter  $\overline{ab}, c\tilde{d}$  sitam.
- $R$  • angulum rectum.
- $ab \triangleq cd$  •  $cab = acd$ .
- $\equiv$  • congruens.\*
- $x \curvearrowright a$  •  $x$  tendere ad limitem  $a$ .
- $\bigcirc r$  • peripheriam circuli radii  $r$ .
- $\odot r$  • aream circuli radii  $r$ .

\* Sit fas, signo hocce, quo summus Geometra GAUSS numeros congruos insignivit, congruentiam geometricam quoque denotare: nulla ambiguitate exinde metuenda.



## §. 1.

Si rectam  $am$  non secet plani eiusdem recta  $bñ$ , at secet quævis  $b\tilde{p}$  Fig. 1.  
(in  $abn$ ): designetur hoc per

$$bn \parallel am.$$

*Dari* talem  $bñ$ , et quidem *unicam*, e quovis puncto  $b$  (extra  $\overline{am}$ ),  
atque

$$bam + abn \text{ non } > 2R$$

esse patet; nam  $bc$  circa  $b$  mota, donec

$$bam + abc = 2R$$

fiat,  $b\tilde{c}$  aliquando *primo* non secat  $am$ , estque tunc  $bc \parallel am$ .

Nec non patet esse  $bn \parallel em$ , ubivis sit  $e$  in  $\overline{am}$  (supponendo in omnibus talibus casibus esse  $am > ae$ ).

Et si, puncto  $c$  in  $am$  abeunte in infinitum, semper sit  $c\tilde{d} = cb$ : erit semper

$$c\tilde{d}b = (cb\tilde{d} < nbc);$$

ast  $nbc \rightarrow 0$ ; adeoque et  $adb \rightarrow 0$ .

## §. 2.

*Si*  $bn \parallel am$ ; *est quoque*  $cn \parallel am$ .

Fig. 2.

Nam sit  $d$  ubicunque in  $macn$ . Si  $c$  in  $bñ$  sit;  $b\tilde{d}$  secat  $am$  (propter  $bn \parallel am$ ), adeoque et  $c\tilde{d}$  secat  $am$ ; si vero  $c$  in  $b\tilde{p}$  fuerit; sit  $bq \parallel c\tilde{d}$ : cadit  $b\tilde{q}$  in  $abn$  (§. 1.) secatque  $am$ , adeoque et  $c\tilde{d}$  secat  $am$ . Quævis  $c\tilde{d}$  igitur (in  $acn$ ) secat in utroque casu  $am$  absque eo, ut  $cñ$  ipsam  $am$  secet. Est ergo semper  $cn \parallel am$ .



## §. 3.

Fig. 2. *Si tam  $br$  quam  $cs$  sit  $\parallel am$ , et  $c$  non sit in  $\overline{br}$ ; tum  $b\overline{r}$ ,  $c\overline{s}$  se invicem haud secant.*

Si enim  $b\overline{r}$ ,  $c\overline{s}$  punctum  $d$  commune haberent; (per §. 2.) essent  $dr$  et  $ds$  simul  $\parallel am$ , caderetque (§. 1.)  $d\overline{s}$  in  $d\overline{r}$  et  $c$  in  $\overline{br}$  (contra hyp.).

## §. 4.

Fig. 3. *Si  $man > mab$ ; pro quovis puncto  $b$  ipsius  $a\overline{b}$  datur tale  $c$  in  $a\overline{m}$ , ut sit  $bcm = nam$ .*

Nam datur (per §. 1.)  $b\overline{dm} > nam$ , adeoque  $m\overline{dp} = man$ , caditque  $b$  in  $n\overline{adp}$ . Si igitur  $nam$  iuxta  $am$  feratur, usquequo  $a\overline{m}$  in  $d\overline{p}$  veniat; aliquando  $a\overline{m}$  per  $b$  transiisse, et aliquod  $bcm = nam$  esse oportet.

## §. 5.

Fig. 1. *Si  $bn \parallel am$ , datur tale punctum  $f$  in  $a\overline{m}$ , ut sit  $f\overline{m} \triangleq bn$ .*

Nam (per §. 1.) datur  $bcm > cbn$ ; et si  $ce = cb$ , adeoque  $ec \triangleq bc$ ; patet esse  $bem < ebn$ . Feratur  $p$  per  $ec$ , angulo  $bpm$  semper  $u$ , et angulo  $pbn$  semper  $v$  dicto; patet  $u$  esse prius ei simultaneo  $v$  minus, posterius vero esse maius. Crescit vero  $u$  a  $bem$  usque  $bcm$  continuo; cum (per §. 4.) nullus angulus  $> bem$  et  $< bcm$  detur, cui  $u$  aliquando  $=$  non fiat; pariter decrescit  $v$  ab  $ebn$  usque  $cbn$  continuo: datur itaque in  $ec$  tale  $f$ , ut  $b\overline{fm} = f\overline{bn}$  sit.

## §. 6.

*Si  $bn \parallel am$ , atque ubivis sit  $e$  in  $a\overline{m}$  et  $g$  in  $b\overline{n}$ : tum  $gn \parallel em$  et  $em \parallel gn$ .*

Nam (per §. 1.) est  $bn \parallel em$ , et hinc (per §. 2.)  $gn \parallel em$ .

Si porro  $f\overline{m} \triangleq bn$  (§. 5.); tum  $m\overline{f}bn = n\overline{b}fm$ , adeoque (cum  $bn \parallel f\overline{m}$  sit) etiam  $f\overline{m} \parallel bn$ , et (per præc.)  $em \parallel gn$ .

## §. 7.

*Si tam  $bn$  quam  $cp$  sit  $\parallel am$ , et  $c$  non sit in  $\overline{bn}$ : est etiam  $bn \parallel cp$ .* Fig. 4.

Nam  $b\overline{n}$ ,  $c\overline{p}$  se invicem non secant (§. 3.); sunt vero  $am$ ,  $bn$ ,  $cp$  aut in plano, aut non; atque in casu primo  $am$  aut in  $bncp$  est, aut non.

Si  $am$ ,  $bn$ ,  $cp$  in plano sint, et  $am$  in  $bncp$  cadat; tum quævis  $b\overline{q}$  (in  $nbc$ ) secat  $\overline{am}$  in aliquo puncto  $d$  (quia  $bn \parallel am$ ); porro cum  $dm \parallel cp$  sit (§. 6.), patet  $d\overline{q}$  secare  $c\overline{p}$ , adeoque esse  $bn \parallel cp$ .

Si vero  $bn$ ,  $cp$  in eadem plaga ipsius  $am$  sint; tum aliqua earum ex. gr.  $cp$  intra duas reliquas  $\overline{bn}$ ,  $\overline{am}$  cadit; quævis  $b\overline{q}$  (in  $nba$ ) autem secat  $\overline{am}$ , adeoque et ipsam  $c\overline{p}$ . Est itaque  $bn \parallel cp$ .

Si  $mab$ ,  $mac$  *angulum* efficiant: tum  $c\overline{bn}$  cum  $a\overline{bn}$  nonnisi  $b\overline{n}$ ,  $a\overline{m}$  vero (in  $abn$ ) cum  $b\overline{n}$ , adeoque  $nbc$  quoque cum  $a\overline{m}$ , nihil commune habent. Per quamvis  $b\overline{d}$  (in  $nba$ ) autem positum  $b\overline{c}\overline{d}$  secat  $a\overline{m}$ , quia (propter  $bn \parallel am$ )  $b\overline{d}$  secat  $a\overline{m}$ . Moto itaque  $b\overline{c}\overline{d}$  circa  $bc$ , donec ipsam  $a\overline{m}$  *prima vice* deserat, postremo cadet  $b\overline{c}\overline{d}$  in  $b\overline{c}\overline{n}$ . Eadem ratione cadet idem in  $b\overline{c}\overline{p}$ ; cadit igitur  $bn$  in  $bcp$ . Porro si  $br \parallel cp$ ; tum (quia etiam  $am \parallel cp$ ) pari ratione cadit  $br$  in  $bam$ ; nec non (propter  $br \parallel cp$ ) in  $bcp$ . Itaque  $b\overline{r}$  ipsis  $mab$ ,  $pcb$  commune, nempe ipsum  $b\overline{n}$  est, atque hinc  $bn \parallel cp$ .

Si igitur  $cp \parallel am$ , et  $b$  extra  $\overline{cam}$  sit: tum sectio ipsorum  $bam$ ,  $bcp$ , nempe  $b\overline{n}$  est  $\parallel$  tam ad  $am$ , quam ad  $cp$ .\*

## §. 8.

*Si  $bn \parallel$  et  $\simeq cp$  (vel brevius  $bn \parallel \simeq cp$ ), atque  $am$  (in  $nbc$ ) *rectam* Fig. 5.  
 $bc$  perpendiculariter bisecet; tum  $bn \parallel am$ .*

Si enim  $b\overline{n}$  secaret  $a\overline{m}$ , etiam  $c\overline{p}$  secaret  $a\overline{m}$  in eodem puncto (cum  $mabn \simeq macp$ ), quod et ipsis  $b\overline{n}$ ,  $c\overline{p}$  commune esset, quamvis  $bn \parallel cp$  sit. Quævis  $b\overline{q}$  (in  $c\overline{bn}$ ) vero secat  $c\overline{p}$ ; adeoque secat  $b\overline{q}$  etiam  $a\overline{m}$ . Consequenter  $bn \parallel am$ .

\* Casu tertio *praemisso* duo priores, adinstar casus secundi §. 10. brevius ac elegantius simul absolvi possunt. (Ed. I. Tom. I. Errata Appendicis).

## §. 9.

Fig. 6. Si  $bn \parallel am$ ,  $map \perp mab$ , atque *angulus, quem*  $nbd$  cum  $nba$  (in ea plaga ipsius  $mabn$ , ubi  $map$  est) *facit, sit*  $< R$ : tum  $map$  et  $nbd$  *se invicem secant*.

Nam sit

$$bam = R, \text{ ac } \perp bn$$

(sive in  $b$  cadat  $c$ , sive non), et

$$ce \perp bn \text{ (in } nbd);$$

erit (per hyp.)  $ace < R$ , et  $af (\perp ce)$  in  $ace$  cadet. Sit  $a\tilde{p}$  sectio (punctum a commune habentium)  $ab\tilde{f}$  et  $am\tilde{p}$ ; erit

$$bap = bam = R$$

(cum sit  $bam \perp map$ ). Si denique  $ab\tilde{f}$  in  $abm$  ponatur ( $a$  et  $b$  manentibus); cadet  $a\tilde{p}$  in  $am$ ; atque cum

$$ac \perp bn \text{ et } af < ac$$

sit, patet  $af$  *intra*  $bfi$  terminari, adeoque  $bf$  in  $abn$  cadere. Secat autem  $b\tilde{f}$  ipsam  $a\tilde{p}$  in *hoc* situ (quia  $bn \parallel am$ ), adeoque etiam in situ *primo*  $a\tilde{p}$  et  $b\tilde{f}$  se invicem secant; estque punctum sectionis ipsis  $map$  et  $nbd$  commune: secant itaque  $map$  et  $nbd$  se invicem.

Facile exhinc sequitur  $map$  et  $nbd$  se mutuo secare, si summa interiorum, quos cum  $mabn$  efficiunt,  $< 2R$  sit.

## §. 10.

Fig. 7. Si tam  $bn$  quam  $cp$  sit  $\parallel \simeq am$ ; est etiam  $bn \parallel \simeq cp$ .

Nam  $mab$  et  $mac$  aut *angulum* efficiunt, aut in plano sunt.

Si prius; bisecet  $\overline{qdf}$  rectam  $ab$  perpendiculariter; erit  $dq \perp ab$ , adeoque  $dq \parallel am$  (§. 8.); pariter si  $\overline{ers}$  bisecet rectam  $ac$  perpendiculariter, est  $er \parallel am$ ; unde  $dq \parallel er$  (§. 7.). Facile hinc (per §. 9.) consequitur,  $\overline{qdf}$

et  $\overline{ef}$  se mutuo secare, et sectionem  $\overline{fs}$  esse  $\parallel \delta q$  (§. 7.), atque (propter  $bn \parallel \delta q$ ) esse etiam

$$\overline{fs} \parallel \overline{bn}.$$

Est porro (pro quovis puncto ipsius  $\overline{fs}$ )

$$fb = fa = fc,$$

caditque  $\overline{fs}$  in planum  $\overline{tgf}$ , rectam  $bc$  perpendiculariter bisecans. Est vero (per §. 7.) (cum sit  $\overline{fs} \parallel \overline{bn}$ ) etiam

$$gt \parallel \overline{bn}.$$

Pari modo demonstratur  $gt \parallel cp$  esse. Interim  $gt$  bisecat rectam  $bc$  perpendiculariter; adeoque  $tgbn = tgcp$  (§. 1.) et

$$bn \parallel \triangle cp.$$

Si  $bn$ ,  $am$ ,  $cp$  in plano sint; sit (*extra hoc planum cadens*)  $\overline{fs} \parallel \triangle am$ ; tum (per præc.)  $\overline{fs} \parallel \triangle$  tam ad  $bn$  quam ad  $cp$ , adeoque et  $bn \parallel \triangle cp$ .

#### §. 11.

Complexus puncti  $a$ , atque *omnium* punctorum, quorum quodvis  $b$  tale est, ut si  $bn \parallel am$  sit, sit etiam  $bn \triangle am$ ; dicatur  $F$ : sectio vero ipsius  $F$  cum quovis plano rectam  $am$  complectente nominetur  $L$ .

In quavis recta, quæ  $\parallel am$  est,  $F$  gaudet puncto, et nonnisi uno; atque patet  $L$  per  $am$  dividi in duas partes congruentes; dicatur  $am$  *axis* ipsius  $L$ ; patet etiam, in quovis plano rectam  $am$  complectente, pro *axe*  $am$  unicum  $L$  dari. Quodvis eiusmodi  $L$ , dicatur  $L$  *ipsius*  $am$  (in plano, de quo agitur, intelligendo). Patet per  $L$  circa  $am$  revolutum,  $F$  describi, cuius  $am$  *axis* vocetur, et vicissim  $F$  *axi*  $am$  *attribuatur*.

#### §. 12.

Si  $b$  ubivis in  $L$  ipsius  $am$  fuerit, et  $bn \parallel \triangle am$  (§. 11.); tum  $L$  ipsius  $am$  et  $L$  ipsius  $bn$  coincidunt.

Nam dicatur  $L$  ipsius  $hñ$  distinctionis ergo  $l$ : sitque  $c$  ubivis in  $L$  et  $cp = bn$  §. 11.: erit cum et  $bn = am$  sit  $cp = am$  §. 12., adeoque  $c$  etiam in  $L$  cadet. Et si  $c$  ubivis in  $L$  sit, et  $cp = am$ : tum  $cp = bn$  §. 12.; caditque  $c$  etiam in  $l$  §. 11.. Itaque  $L$  et  $l$  sunt eadem: ac quævis  $hñ$  est etiam axis ipsius  $L$ , et inter omnes axes ipsius  $L$ ,  $=$  est. Idem de  $F$  eodem modo patet.

## §. 13.

Fig. 2. Si  $bn = am$ ,  $cp = dq$ , et  $bam + abn = 2R$  sit; tum etiam  $dcp + cdq = 2R$ . Sit enim  $ea = eb$  et  $efm = dcp$  §. 4.: erit cum

$$bam + abn = 2R = abn + abg$$

sit

$$ebg = eaf;$$

adeoque si etiam  $bg = af$  sit,

$$\triangle ebg = \triangle eaf, \quad beg = aef,$$

cadetque  $g$  in  $fē$ . Est porro  $gfm + fgn = 2R$  quia  $egb = efa$ . Est etiam  $gn = fm$  §. 6.; itaque si  $mfrs = pcq$ , tum  $rs = gn$  §. 7., et  $r$  in vel extra  $fg$  cadit (si  $cd$  non  $= fg$ , ubi res iam patet).

I. In casu primo est  $f rs$  non  $> \frac{1}{2}2R - rfm = fgn$ , quia  $rs = fm$ ; ast cum  $rs = gn$  sit, est etiam  $f rs$  non  $< fgn$ ; adeoque  $f rs = fgn$ , et

$$rfm + f rs = gfm + fgn = 2R.$$

Itaque et  $dcp + cdq = 2R$ .

II. Si  $r$  extra  $fg$  cadat; tunc  $ngr = mfr$ , sitque  $mfgn = nghl = lhfo$  et ita porro, usquequo  $fl =$  vel prima vice  $> fr$  fiat. Est heic  $fo = hl = fm$  (§. 7.). Si  $f$  in  $r$  cadat; tum  $fo$  in  $rs$  cadit (§. 1.); adeoque

$$rfm + f rs = f fm + f fo = f fm + fgn = 2R;$$

si vero  $r$  in  $hl$  cadat, tum (per I.) est

$$rhl + hrs = 2R = rfm + f rs = dcp + cdq.$$

## §. 14.

*Si*  $bn \parallel am$ ,  $cp \parallel dq$ , *et*  $bam + abn < 2R$  *sit*; *tum etiam*  $dcp + cdq < 2R$ .

*Si enim*  $dcp + cdq$  *non esset*  $<$ , adeoque (per §. 1.) *esset*  $= 2R$ ; *tum* (per §. 13.) *etiam*  $bam + abn = 2R$  *esset* (contra hyp.).

## §. 15.

Perpensis §§. 13. et 14. *Systema Geometriae hypotesi veritatis Axiomatis Euclidei XI. insistens dicatur*  $\Sigma$ ; *et hypotesi contrariae superstructum sit*  $S$ . *Omnia, quae expresse non dicentur, in*  $\Sigma$  *vel in*  $S$  *esse; absolute enuntiari, i. e. illa, sive*  $\Sigma$  *sive*  $S$  *reipsa sit, vera asseri intelligatur.*

## §. 16.

*Si*  $am$  *sit axis alicuius*  $L$ ; *tum*  $L$  *in*  $\Sigma$  *recta*  $\perp am$  *est.*

Fig. 5.

*Nam sit*  $e$  *quovis puncto*  $b$  *ipsius*  $L$  *axis*  $bn$ ; *erit in*  $\Sigma$

$$bam + abn = 2bam = 2R,$$

adeoque  $bam = R$ . *Et si*  $c$  *quodvis punctum in*  $\overline{ab}$  *sit, atque*  $cp \parallel am$ ; *est* (per §. 13.)  $cp \simeq am$ , adeoque  $c$  *in*  $L$  (§. 11.).

*In*  $S$  *vero nulla 3 puncta*  $a, b, c$  *ipsius*  $L$  *vel*  $F$  *in recta sunt.*

*Nam aliquis axium*  $am, bn, cp$  (ex. gr.  $am$ ) *intra duos reliquos cadit*; *et tunc* (per §. 14.) *tam*  $bam$  *quam*  $cam < R$ .

## §. 17.

*L est etiam in*  $S$  *linea, et*  $F$  *superficies.*

Fig. 7.

*Nam* (per §. 11.) *quodvis planum ad axem*  $am$  (per punctum aliquod ipsius  $F$ ) *perpendiculare secat ipsum*  $F$  *in peripheria circuli, cuius planum* (per §. 14.) *ad nullum alium axem*  $bñ$  *perpendiculare est. Revolvatur*  $F$  *circa*  $bn$ ; *manebit* (per §. 12.) *quodvis punctum ipsius*  $F$  *in*  $F$ , *et sectio ipsius*  $F$  *cum plano ad*  $bñ$  *non perpendiculari describet super-*



*quivis angulus Llineus in F angulo planorum ad F per crura perpendicularium aequalis est.*

## §. 21.

*Duae lineae Lformes  $a\tilde{p}$ ,  $b\tilde{d}$  in eodem F, cum tertia Lformi  $ab$  summam internorum  $< 2R$  efficientes, se mutuo secant* (per  $a\tilde{p}$  in F intelligendo L per a, p ductum, per  $a\tilde{p}$  vero dimidium illud eius ex a incipiens, in quod p cadit). Fig. 6.

Nam si  $am$ ,  $bn$  axes ipsius F sint; tum  $am\tilde{p}$ ,  $bn\tilde{d}$  secant se invicem (§. 9.); atque F secat eorundem sectionem (per §§. 7. et 11.); adeoque et  $a\tilde{p}$ ,  $b\tilde{d}$  se mutuo secant.

Patet exhinc Axioma XI. et omnia, quæ in Geometria Trigonometriaque (plana) asseruntur, *absolute* constare in F, rectarum vices lineis L subeuntibus: idcirco functiones trigonometricæ abhinc eodem sensu accipientur, quo in  $\Sigma'$  veniunt; et periphæria circuli, cuius radius Lformis  $= r$  in F, est  $= 2\pi r$ , et pariter  $\odot r$  (in F)  $= \pi r^2$  (per  $\pi$  intelligendo  $\frac{1}{2} \odot 1$  in F, sive notum 3.1415926 . . .).

## §. 22.

*Si  $a\tilde{b}$  fuerit L ipsius  $a\tilde{m}$ , et c in  $a\tilde{m}$ ; atque angulus  $cab$  (e recta  $a\tilde{m}$  et Lformi linea  $a\tilde{b}$  compositus) feratur prius iuxta  $a\tilde{b}$ , tum iuxta  $b\tilde{a}$  semper porro in infinitum: erit via  $c\tilde{d}$  ipsius c linea L ipsius  $c\tilde{m}$ .* Fig. 9.

Nam (posteriore l dicta) sit punctum quodvis  $d$  in  $c\tilde{d}$ ,  $dn \parallel cm$ , et b punctum ipsius L in  $\tilde{dn}$  cadens; erit  $bn \triangleq am$ , et  $ac = bd$ , adeoque  $dn \triangleq cm$ , consequ.  $d$  in l. Si vero  $d$  in l et  $dn \parallel cm$ , atque b punctum ipsius L ipsi  $\tilde{dn}$  commune sit; erit  $am \triangleq bn$  et  $cm \triangleq dn$ , unde manifesto  $bd = ac$ , cadetque  $d$  in viam puncti c, et sunt l et  $c\tilde{d}$  eadem. Designetur tale l per l  $\parallel L$ .

## §. 23.

Si linea Lformis  $c\tilde{d}f \parallel abe$  (§. 22.), et  $ab = be$ , atque  $a\tilde{m}$ ,  $b\tilde{n}$ ,  $e\tilde{p}$  sint axes; erit manifesto  $c\tilde{d} = d\tilde{f}$ ; et si quælibet 3 puncta a, b, e fuerint ipsius Fig. 9.



1. The first part of the paper is devoted to the study of the properties of the function  $f(x)$  defined by the equation

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

where  $a_n$  are the coefficients of the power series. It is shown that the function  $f(x)$  is analytic in the whole plane and that it satisfies the differential equation

$$f'(x) = f(x) + x f''(x)$$

where  $f'(x)$  and  $f''(x)$  are the first and second derivatives of  $f(x)$  respectively.

$$f(0) = 1$$

$$f'(0) = 0$$

It is also shown that the function  $f(x)$  is convex for  $x > 0$ .

$$f''(x) = f(x) + x f'''(x)$$

where

$$f'''(x) = f(x) + x f^{(4)}(x)$$

and

$$f^{(4)}(x) = f(x) + x f^{(5)}(x)$$

It is also shown that the function  $f(x)$  is the generating function of the sequence  $a_n$  defined by the recurrence relation

$$a_{n+1} = a_n + n a_{n-1}$$

unde  $\text{ambn} \equiv \text{amep}$  erit, etsi hoc illius qualevis multipulum sit; quod singulare quidem est, sed absurditatem ipsius  $S$  evidenter non probat.

## §. 25.

*In quovis rectilineo triangulo sunt peripheriae radiorum lateribus aequalium, uti sinus angulorum oppositorum.* Fig. 10.

Sit enim  $\text{abc} = R$ , et  $\text{am} \perp \text{bac}$ , atque sint  $\text{bn}$ ,  $\text{cp} \parallel \text{am}$ ; erit  $\text{cab} \perp \text{ambn}$ , adeoque (cum  $\text{cb} \perp \text{ba}$  sit)  $\text{cb} \perp \text{ambn}$ , consequ.  $\text{cpbn} \perp \text{ambn}$ . Secet  $F$  ipsius  $\text{cp}$  rectas  $\text{bn}$ ,  $\text{am}$  (respective) in  $\text{d}$ ,  $\text{e}$ , et fascias  $\text{cpbn}$ ,  $\text{cpam}$ ,  $\text{bnam}$  in lineis  $L$ formibus  $\text{cd}$ ,  $\text{ce}$ ,  $\text{de}$ ; erit (§. 20.)  $\text{cde} = \text{angulo ipsorum } \text{ndc}$ ,  $\text{nde}$ , adeoque  $= R$ ; atque pari ratione est  $\text{ced} = \text{cab}$ .

Est autem (per §. 21.) in  $L$ lineo triangulo  $\text{ced}$  (heic radio semper  $= 1$  posito)

$$\text{ec} : \text{dc} = 1 : \sin. \text{dec} = 1 : \sin. \text{cab}.$$

Est quoque (per §. 21.)

$$\begin{aligned} \text{ec} : \text{dc} &= \odot \text{ec} : \odot \text{dc} \text{ (in } F) \\ &= \odot \text{ac} : \odot \text{bc} \text{ (§. 18.)}; \end{aligned}$$

adeoque est etiam

$$\odot \text{ac} : \odot \text{bc} = 1 : \sin. \text{cab};$$

unde assertum pro quovis triangulo liquet.

## §. 26.

*In quovis sphaerico triangulo sunt sinus laterum, uti sinus angulorum iisdem oppositorum.* Fig. 11.

Nam sit  $\text{abc} = R$ , et  $\text{ced}$  perpendiculare ad sphæræ radium  $\text{oa}$ ; erit  $\text{ced} \perp \text{aob}$ , et (cum etiam  $\text{boc} \perp \text{boa}$  sit)  $\text{cd} \perp \text{ob}$ . In triangulis  $\text{ceo}$ ,  $\text{cdo}$  vero est (per §. 25.)

$$\begin{aligned} \odot \text{ec} : \odot \text{oc} : \odot \text{dc} &= \sin. \text{coe} : 1 : \sin. \text{cod} \\ &= \sin. \text{ac} : 1 : \sin. \text{bc}; \end{aligned}$$

interim (§. 25.) etiam

itaque  $\odot ec : \odot dc = \sin. cde : \sin. ced;$   
 $\sin. ac : \sin. bc = \sin. cde : \sin. ced;$   
 est vero  $cde = R = cba$ , atque  $ced = cab$ . Consequenter  
 $\sin. ac : \sin. bc = 1 : \sin. a.$

*E quo promanans Trigonometria sphaerica ab Axiomate XI. independenter stabilita est.*

§. 27.

Fig. 12. Si  $ac, bd$  sint  $\perp ab$ , et feratur  $cab$  iuxta  $\overline{ab}$ ; erit (via puncti  $c$  dicta heic  $cd$ )

$$cd : ab = \sin. u : \sin. v.$$

Nam sit  $de \perp ca$ ; est in triangulis  $ade, adb$  (per §. 25.)

$$\odot ed : \odot ad : \odot ab = \sin. u : 1 : \sin. v.$$

Revoluto  $bacd$  circa  $ac$ , describetur  $\odot ab$  per  $b$ ,  $\odot ed$  per  $d$ ; et via dictæ  $cd$  denotetur heic per  $\odot cd$ . Sit porro polygonum quodvis  $bfg \dots$  ipsi  $\odot ab$  inscriptum; nascetur per plana ex omnibus lateribus  $bf, fg \&$ , ad  $\odot ab$  perpendicularia, in  $\odot cd$  quoque figura polygonalis totidem laterum; et demonstrari (ad instar §. 23.) potest, esse

$$cd : ab = dh : bf = hf : fg = \dots,$$

adeoque

$$dh + hf + \dots : bf + fg + \dots = cd : ab$$

Quovis laterum  $bf, fg, \dots$  ad limitem  $o$  tendente, manifesto

$$bf + fg + \dots \curvearrowright \odot ab \quad \text{et} \quad dh + hf + \dots \curvearrowright \odot ed.$$

Itaque etiam

$$\odot ed : \odot ab = cd : ab.$$

Erat vero

$$\odot ed : \odot ab = \sin. u : \sin. v.$$

Consequ.

$$cd : ab = \sin. u : \sin. v.$$

Remoto ac a  $bd$  in infinitum, manet

$$\begin{aligned} & cd : ab \\ \text{adeoque etiam} & \sin. u : \sin. v \end{aligned}$$

*constans*;  $u$  vero  $\sphericalangle R$  (§. 1.), et si  $dm \parallel bn$  sit,  $v \sphericalangle z$ ; unde fit

$$cd : ab = 1 : \sin. z.$$

Via dicta  $cd$  denotabitur per  $cd \parallel ab$ .

### §. 28.

*Si*  $bn \parallel \sphericalangle am$ , *et*  $c$  *in*  $am$ , *atque*  $ac = x$  *sit: erit*  $X$  (§. 23.)

Fig. 13.

$$= \sin. u : \sin. v.$$

Nam si  $cd$  et  $ae$  sint  $\perp bn$  et  $bf \perp am$ ; erit (ad instar §. 27.)

$$\odot bf : \odot cd = \sin. u : \sin. v.$$

Est autem evidenter  $bf = ae$ : quamobrem

$$\odot ea : \odot dc = \sin. u : \sin. v.$$

In superficiebus vero  $F$ formibus ipsorum  $am$  et  $cm$  (ipsum  $ambn$  in  $ab$  et  $cq$  secantibus) est (per §. 21.)

$$\odot ea : \odot dc = ab : cq = X.$$

Est itaque etiam

$$X = \sin. u : \sin. v.$$

### §. 29.

*Si*  $bam = R$ ,  $ab = y$ , *et*  $bn \parallel am$  *sit; erit in*  $S$

Fig. 14.

$$Y = \cot. \frac{1}{2} u.$$

Nam si fuerit  $ab=ac$ , et  $cp \parallel am$  (adeoque  $bn \parallel \triangle cp$ ), atque  $pcd=qcd$ ; datur (§. 19.)  $\delta s \perp c\tilde{d}$ , ut  $\delta s \perp cp$ , adeoque (§. 1.)  $\delta t \parallel cq$  sit. Si porro  $be \perp \delta s$ ; erit (§. 7.)  $\delta s \parallel bn$ , adeoque (§. 6.)  $bn \parallel es$ , et (cum  $\delta t \parallel cq$  sit)  $bq \parallel et$ ; consequ. (§. 1.)  $ebn=ebq$ .

Repræsententur,  $bcf$  ex  $L$  ipsius  $bn$ , et  $fg, \delta h, cf$  et  $el$  ex  $L$  formibus lineis ipsorum  $\tilde{f}t, \delta t, cq$  et  $et$ ; erit evidenter (§. 22.)

$$\text{itaque} \quad hg = \delta f = \delta h = hc;$$

$$\text{Pariter patet} \quad cq = 2ch = 2v.$$

$$\text{esse. Est vero} \quad bg = 2bl = 2z$$

$$\text{quapropter} \quad bc = bg - cq;$$

$$\text{adeoque (§. 24.)} \quad y = z - v,$$

$$\text{Est demum (§. 28.)} \quad Y = Z : I'.$$

$$Z = 1 : \sin. \frac{1}{2} u \quad \text{et} \quad I' = 1 : \sin. \left( R - \frac{1}{2} u \right),$$

consequ.

$$Y = \cot. \frac{1}{2} u.$$

### §. 30.

Fig. 15. Verumtamen facile (ex §. 25.) patet, resolutionem problematis *Trigonometriae planae* in  $S$ , peripheriæ per radium expressæ indigere; hoc vero rectificatione ipsius  $L$  obtineri potest.

Sint  $ab, cm, c'm' \perp a\tilde{c}$ , atque  $b$  ubivis in  $a\tilde{b}$ ; erit (§. 25.)

$$\text{et} \quad \sin. u : \sin. v = \odot p : \odot y$$

$$\text{adeoque} \quad \sin. u' : \sin. v' = \odot p : \odot y';$$

$$\frac{\sin. u}{\sin. v} \odot = \frac{\sin. u'}{\sin. v'} \odot y'.$$

Est vero per §. 27.

consequ.  $\sin. v : \sin. v' = \cos. u : \cos. u';$

$$\frac{\sin. u}{\cos. u} \circ y = \frac{\sin. u'}{\cos. u'} \circ y',$$

seu

$$\circ y : \circ y' = \tan. u' : \tan. u = \tan. w : \tan. w'.$$

Sint porro  $cn \parallel ab$ ,  $c'n' \parallel ab$  et  $cd$ ,  $c'd'$  lineæ  $L$  formes ad  $\overline{ab}$  perpendiculares; erit (§. 21.) etiam

$$\circ y : \circ y' = r : r',$$

adeoque

$$r : r' = \tan. w : \tan. w'.$$

Crescat iam  $p$  ab  $a$  incipiendo in infinitum; tum  $w \frown z$  et  $w' \frown z'$ ; quapropter etiam

$$r : r' = \tan. z : \tan. z'.$$

*Constans*  $r : \tan. z$  (ab  $r$  *independens*) dicatur  $i$ ; dum  $y \frown 0$ , est

$$\left( \frac{r}{y} = \frac{i \tan. z}{y} \right) \frown 1,$$

adeoque

$$\frac{y}{\tan. z} \frown i.$$

Ex §. 29. fit

$$\tan. z = \frac{1}{2} (Y - Y^{-1});$$

itaque

$$\frac{2y}{Y - Y^{-1}} \frown i,$$

seu (§. 24.)

$$\frac{2y I^{\frac{y}{i}}}{I^{\frac{2y}{i}} - 1} \frown i.$$

Notum autem est, expressionis istius (dum  $y \frown 0$ ) limitem esse  $\frac{i}{\log. \text{nat. } I}$ ; est ergo

$$\frac{i}{\log. \text{nat. } I} = i \quad \text{et} \quad I = e = 2.7182818 \dots,$$

quæ quantitas insignis hic quoque elucet. Si nempe abhinc  $i$  illam rectam denotet, cuius  $I=e$  sit, erit  $r=i \text{ tang. } z$ . Erat autem (§. 21.)  $\bigcirc y=2\pi r$ ; est igitur

$$\begin{aligned}\bigcirc y &= 2\pi i \text{ tang. } z = \pi i (Y - Y^{-1}) = \\ &= \pi i (e^{\frac{y}{i}} - e^{-\frac{y}{i}}) = \frac{\pi y}{\log. \text{ nat. } Y} (Y - Y^{-1})\end{aligned}$$

(per §. 24.).

### §. 31.

Fig. 16. Ad resolutionem omnium triangulorum rectangulorum rectilineorum trigonometricam (e qua omnium triangulorum resolutio in promptu est) in  $S$  3 æquationes sufficiunt: nempe ( $a, b$  cathetos,  $c$  hypotenusam, et  $\alpha, \beta$  angulos cathetis oppositos denotantibus) æquatio relationem exprimens *primo* inter  $a, c, \alpha$ , *secundo* inter  $a, \alpha, \beta$ , *tertio* inter  $a, b, c$ ; nimirum ex his *reliquae* 3 per eliminationem prodeunt.

I. Ex §§. 25. et 30. est

$$1 : \sin. \alpha = (C - C^{-1}) : (A - A^{-1}) = (e^{\frac{c}{i}} - e^{-\frac{c}{i}}) : (e^{\frac{a}{i}} - e^{-\frac{a}{i}});$$

(æquatio pro  $\alpha, c, a$ ).

II. Ex §. 27. sequitur (si  $\beta m \parallel \gamma n$  sit)

$$\cos. \alpha : \sin. \beta = 1 : \sin. u;$$

ex §. 29. autem fit

$$1 : \sin. u = \frac{1}{2} (A + A^{-1});$$

itaque

$$\cos. \alpha : \sin. \beta = \frac{1}{2} (A + A^{-1}) = \frac{1}{2} (e^{\frac{a}{i}} + e^{-\frac{a}{i}});$$

(æquatio pro  $\alpha, \beta, a$ ).

III. Si  $\alpha\alpha' \perp \beta\alpha\gamma$ , atque  $\beta\beta'$  et  $\gamma\gamma'$  fuerint  $\parallel \alpha\alpha'$ , (§. 27.), atque  $\beta'\alpha'\gamma' \perp \alpha\alpha'$ ; erit manifesto (uti in §. 27.)

$$\begin{aligned}\frac{\beta\beta'}{\gamma\gamma'} &= \frac{1}{\sin. u} = \frac{1}{2} (A + A^{-1}), \\ \frac{\gamma\gamma'}{\alpha\alpha'} &= \frac{1}{2} (B + B^{-1}),\end{aligned}$$

ac

$$\frac{\beta\beta'}{\alpha\alpha'} = \frac{1}{2}(C + C^{-1});$$

consequ.

$$\frac{1}{2}(C + C^{-1}) = \frac{1}{2}(A + A^{-1}) \frac{1}{2}(B + B^{-1}),$$

sive

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2}(e^{\frac{a}{i}} + e^{-\frac{a}{i}})(e^{\frac{b}{i}} + e^{-\frac{b}{i}});$$

(æquatio pro  $a, b, c$ ).Si  $\gamma\alpha\delta = R$ , et  $\beta\delta \perp \alpha\delta$  sit; erit

$$\odot c : \odot a = 1 : \sin. \alpha,$$

et

$$\odot c : \odot (d = \beta\delta) = 1 : \cos. \alpha,$$

adeoque ( $\odot x^2$  pro quovis  $x$  factum  $\odot x . \odot x$  denotante) manifesto

$$\odot a^2 + \odot d^2 = \odot c^2.$$

Est vero (per §. 27. et II.)

$$\odot d = \odot b . \frac{1}{2}(A + A^{-1}),$$

consequ.

$$(e^{\frac{c}{i}} - e^{-\frac{c}{i}})^2 = \frac{1}{4}(e^{\frac{a}{i}} + e^{-\frac{a}{i}})^2(e^{\frac{b}{i}} - e^{-\frac{b}{i}})^2 + (e^{\frac{a}{i}} - e^{-\frac{a}{i}})^2;$$

*alia æquatio pro  $a, b, c$  (cuius membrum secundum facile ad formam symmetricam seu invariabilem reducitur).*

Denique ex

$$\frac{\cos. \alpha}{\sin. \beta} = \frac{1}{2}(A + A^{-1})$$

atque

$$\frac{\cos. \beta}{\sin. \alpha} = \frac{1}{2}(B + B^{-1})$$

fit (per III.)

$$\cot. \alpha \cot. \beta = \frac{1}{2}(e^{\frac{c}{i}} + e^{-\frac{c}{i}});$$

(æquatio pro  $\alpha, \beta, c$ ).



## §. 32.

Restat adhuc modum *problemata* in  $S$  resolvendi breviter ostendere, quo (per exempla magis obvia) peracto, demum quid theoria hæcce præstet, candide dicetur.

Fig. 17. I. Sit  $\widetilde{ab}$  linea in plano, et  $y=f(x)$  æquatio eius (pro coordinatis perpendicularibus), et quodvis incrementum ipsius  $z$  dicatur  $dz$ , atque incrementa ipsorum  $x$ ,  $y$ , et areæ  $u$ , eidem  $dz$  respondentia, respective per  $dx$ ,  $dy$ ,  $du$  denotentur; sitque  $bh \parallel cf$ , et exprimatur (ex §§. 31. et 27.)  $\frac{bh}{dx}$  per  $y$ , ac quæraturs ipsius  $\frac{dy}{dx}$  *limes* tendente  $dx$  ad limitem 0, (quod, ubi eiusmodi limes quæritur, subintelligatur): innotescet exinde etiam limes ipsius  $\frac{dy}{bh}$ , adeoque tg.  $hbg$ ; eritque, (cum  $hbc$  manifesto nec  $>$  nec  $<$  adeoque  $= R$  sit), *tangens* in  $b$  ipsius  $bg$  per  $y$  determinata.

II. Demonstrari potest, esse

$$\frac{dz^2}{dy^2 + bh^2} \sim I.$$

Hinc *limes* ipsius  $\frac{dz}{dx}$ , et inde  $z$  integration (per  $x$  expressum) reperitur.

Et potest lineæ cuiusvis *in concreto datae* æquatio in  $S$  inveniri, ex. gr. ipsius  $L$ .

Si enim  $a\bar{m}$  axis ipsius  $L$  sit; tum quævis  $c\bar{b}$  ex  $a\bar{m}$  secat  $L$  (cum per §. 19 quævis recta ex  $a$  præter  $a\bar{m}$  ipsum  $L$  secet); est vero (si  $b\bar{n}$  axis sit)

$$X = 1 : \sin. cbn \quad (§. 28.),$$

atque

$$Y = \cot. \frac{1}{2} cbn \quad (§. 29.),$$

unde fit

$$Y = X + \sqrt{X^2 - 1}$$

seu

$$e^{\frac{y}{i}} = e^{\frac{x}{i}} + \sqrt{e^{\frac{2x}{i}} - 1}$$

æquatio quæsitæ. Erit hinc

$$\frac{dy}{dx} \sim X(X^2-1)^{-\frac{1}{2}};$$

atqui

$$\frac{bh}{dx} = 1 : \sin. cbn = X;$$

adeoque

$$\frac{dy}{bh} \sim (X^2-1)^{-\frac{1}{2}};$$

$$1 + \frac{dy^2}{bh^2} \sim X^2(X^2-1)^{-1},$$

$$\frac{dz^2}{bh^2} \sim X^2(X^2-1)^{-1},$$

$$\frac{dz}{bh} \sim X(X^2-1)^{-\frac{1}{2}}$$

atque

$$\frac{dz}{dx} \sim X^2(X^2-1)^{-\frac{1}{2}};$$

unde per integrationem invenitur

$$z = i(X^2-1)^{\frac{1}{2}} = i \cot. cbn$$

(uti §. 30.).

III. Manifesto

$$\frac{du}{dx} \sim \frac{hfcbh}{dx},$$

quod (nonnisi ab  $y$  dependens) iam primum per  $y$  exprimendum est;  
unde  $u$  integrando prodit.

Si  $ab = p$ ,  $ac = q$ , et  $cd = r$ , atque  $cabdc = s$  sit; poterit (uti in II.) Fig. 12. ostendi esse

$$\frac{ds}{dq} \sim r,$$

quod

$$= \frac{1}{2} p (e^{\frac{q}{i}} + e^{-\frac{q}{i}}),$$

atque integrando

$$s = \frac{1}{2} pi (e^{\frac{q}{i}} - e^{-\frac{q}{i}}).$$



atque (§. 24.)

$$\frac{du}{dx} \sim y,$$

adeoque (integrando)

$$y = re^{-\frac{x}{i}};$$

$$u = ri(1 - e^{-\frac{x}{i}}).$$

Crescente  $x$  in infinitum, fiet in  $S$   $e^{-\frac{x}{i}} \sim 0$ , adeoque  $u \sim ri$ . Per *quantitatem* ipsius  $mabn$  in posterum limes iste intelligetur.

Simili modo invenitur, quod si  $p$  sit figura in  $F$ ; spatium a  $p$  et complexu axium e terminis ipsius  $p$  ductorum clausum  $= \frac{1}{2} pi$  sit.

VI. Si angulus ad centrum segmenti  $z$  sphæræ sit  $2u$ , peripheria Fig. 10. circuli maximi sit  $p$ , et arcus  $fc$  (anguli  $u$ )  $= x$ ; erit (§. 25.)

et hinc

$$1 : \sin. u = p : \bigcirc bc,$$

Interim est

$$\bigcirc bc = p \sin. u.$$

Est porro

$$x = \frac{pu}{2\pi}, \quad \text{ac} \quad dx = \frac{pdu}{2\pi}.$$

et hinc

$$\frac{dz}{dx} \sim \bigcirc bc,$$

unde (integrando)

$$\frac{dz}{du} \sim \frac{p^2}{2\pi} \sin. u,$$

$$z = \frac{\sin. \text{vers. } u}{2\pi} p^2.$$

Cogitetur  $F$  in quod  $p$  (per meditullium  $f$  segmenti transiens) cadit; planis  $\overline{fem}$ ,  $\overline{cem}$  per  $af$ , ac ad  $F$  perpendiculariter positis, ipsumque in  $\overline{feg}$ ,  $ce$  secantibus; et considerentur  $L$  formis  $cd$  (ex  $c$  ad  $\overline{feg}$  perpendicularis) nec non  $L$  formis  $cf$ ; erit (§. 20.)

et (§. 21.)

$$cef = u,$$

$$\frac{\overline{fd}}{p} = \frac{\sin. \text{vers. } u}{2\pi},$$

adeoque

$$z = \text{fd} \cdot p.$$

Ast (§. 21.)

$$p = \pi \cdot \text{fdg},$$

itaque

$$z = \pi \cdot \text{fd} \cdot \text{fdg}.$$

Est autem (§. 21.)

$$\text{fd} \cdot \text{fdg} = \text{fc} \cdot \text{fc};$$

consequ.

$$z = \pi \cdot \text{fc} \cdot \text{fc} = \odot \text{fc} \text{ in } F.$$

Fig. 14. Sit iam  $\text{bj} = \text{cj} = r$ ; erit (§. 30.)

$$2r = i(Y - Y^{-1}),$$

adeoque (§. 21.)

$$\odot 2r \text{ (in } F) = \pi i^2 (Y - Y^{-1})^2.$$

Est quoque (IV.)

$$\odot 2y = \pi i^2 (Y^2 - 2 + Y^{-2});$$

igitur  $\odot 2r \text{ (in } F) = \odot 2y$ , adeoque *et superficies z segmenti sphaerici aequatur circulo, chorda fc tanquam radio descripto.*

Hinc tota sphæræ superficies

$$= \odot \text{fg} = \text{fdg} \cdot p = \frac{p^2}{\pi},$$

*suntque superficies sphaerarum, uti secundae potentiae peripheriarum earundem maximarum.*

VII. Soliditas sphæræ radii  $x$  in  $S$  reperitur simili modo

$$= \frac{1}{2} \pi i^3 (X^2 - X^{-2}) - 2\pi i^2 x;$$

Fig. 12. superficies per revolutionem lineæ cð circa ab orta

$$= \frac{1}{2} \pi i p (Q^2 - Q^{-2}),$$

et corpus per cabðc descriptum

$$= \frac{1}{4} \pi i^2 p (Q - Q^{-1})^2.$$

*Quomodo vero omnia a (IV.) hucusque tractata etiam absque integratione perfici possint, brevitatis studio supprimitur.*

Demonstrari potest, *omnis expressionis literam i continentis* (adeoque *hypothesi*, quod *detur i*, innixæ) *limitem, crescente i in infinitum, exprimere quantitatem plane pro Σ* (adeoque pro *hypothesi nullius i*), *siquidem non eveniant aequationes identicae*. Cave vero intelligas putari, *systema ipsum variari* posse (quod omnino *in se et per se determinatum* est) sed tantum *hypothesein*, quod *successive* fieri potest, donec non ad absurdum perducti fuerimus. *Posito* igitur, quod in *tali* expressione *litera i* pro casu, si *S* esset reipsa, *illam* quantitatem unicam designet, cuius  $I=e$  sit; si vero *revera Σ* fuerit, *limes dictus* loco expressionis accipi *cogitetur*: manifesto *omnes* expressiones ex *hypothesi realitatis* ipsius *S* oriundæ (hoc sensu) *absolute valent*, etsi *prorsus ignotum sit, num Σ sit, aut non sit*.

Ita e. g. ex expressione in §. 30. obtenta facile (et quidem *tam* differentiationis auxilio, quam *absque* eo) valor notus pro  $\Sigma'$  prodit

$$\bigcirc x = 2\pi x;$$

ex I. (§. 31.) rite tractato, sequitur

$$1 : \sin. \alpha = c : a;$$

ex II. vero

$$\frac{\cos. \alpha}{\sin. \beta} = 1, \text{ adeoque } \alpha + \beta = R;$$

æquatio *prima* in III. fit *identica*, adeoque *valet* pro  $\Sigma'$ , quamvis nihil in eo *determinet*; ex *secunda* autem fluit

$$c^2 = a^2 + b^2.$$

*Aequationes notae fundamentales trigonometriae planae* in  $\Sigma'$ .

Porro inveniuntur (ex §. 32.) pro  $\Sigma'$  area et corpus in III., utrumque

$$= pq;$$

ex IV.

$$\odot x = \pi x^2;$$

ex VII. sphæra radii  $x$

$$= \frac{4}{3} \pi x^3$$

§.

Sunt quoque theorematum ad finem VI. enuntiata manifesto *inconditionate vera*.

### §. 33.

Superest adhuc, quid theoria ista sibi velit, in §. 32. promissum exponere.

I. Num  $\Sigma'$  aut  $S$  aliquod *reipsa* sit, indecisum manet.

II. Omnia ex hypothesi *falsitatis* Ax. XI. deducta semper *sensu* §. 32. intelligendo *absolute* valent, adeoque *hoc sensu nulli hypothesi innituntur*. Habetur idcirco *trigonometria plana a priori*, in qua *solum systema verum ignotum* adeoque solummodo *absolutae* magnitudines expressionum incognitæ manent, per *unicum* vero casum notum, manifesto totum systema figeretur. Trigonometria sphærica autem in §. 26. *absolute* stabilitur. Habeturque Geometria, Geometriæ planæ in  $\Sigma$  prorsus analogæ in  $F$ .

III. Si *constaret*  $\Sigma'$  esse, nihil hoc respectu amplius incognitum esset; si vero *constaret non esse*  $\Sigma'$ , tunc (§. 31.) (e. g.) e lateribus  $x, y$  et angulo rectilineo ab iis intercepto, in *concreto datis* manifesto in se et per se impossibile esset triangulum *absolute* resolvere i. e. a priori determinare angulos ceteros et *rationem lateris tertii* ad duo data; nisi  $X, Y$  determinentur, ad quod in *concreto* haberi aliquod *a* oporteret, cuius  $A$  notum esset; atque tum *i unitas naturalis longitudinum* esset, sicuti  $e$  est basis logarithmorum naturalium. Si existentia huius  $i$  constiterit; quomodo ad usum saltem quam exactissime construi possit, ostendetur.

IV. Sensu in I. et II. exposito patet, omnia in spatio methodo recentiorum Analytica intra iustos fines valde laudanda absolvi posse.

V. Denique lectoribus benevolis haud ingratum futurum est; pro casu illo, quodsi non  $\Sigma'$  sed  $S$  reipsa esset, circulo æquale rectilineum construi.

## §. 34.

*Ex  $\delta$  ducitur  $\delta m$  an modo sequente.*

Fig. 12.

Fiat ex  $\delta$

$$\delta b \perp an;$$

erigatur e puncto quovis aliquo a rectæ  $\overline{ab}$

$$ac \perp an \text{ (in } \delta ba),$$

et demittatur

$$\delta e \perp ac;$$

erit (§. 27.)

$$\odot ed : \odot ab = 1 : \sin. z,$$

siquidem fuerit  $\delta m \parallel bn$ .

Est vero  $\sin. z$  non  $> 1$ , adeoque  $ab$  non  $> \delta e$ . Descriptus igitur quadrans radio ipsi  $\delta e$  æquali ex  $a$  in  $bac$ , gaudebit puncto aliquo  $b$  vel  $o$  cum  $b\delta$  communi. Priore in casu manifesto  $z = R$ ; in posteriore vero erit (§. 25.)

$$(\odot ao = \odot ed) : \odot ab = 1 : \sin. aob,$$

adeoque

$$z = aob.$$

Si itaque fiat  $z = aob$ , erit  $\delta m \parallel bn$ .

## §. 35.

Si fuerit  $S$  reipsa; *ducetur recta ad anguli acuti crus unum perpendicularis, quæ ad alterum  $\parallel$  sit, hoc modo.* Fig. 18.

Sit  $am \perp bc$ , et accipiatur  $ab = ac$  tam parvum (per §. 19.), ut si ducatur  $bn \parallel am$  (§. 34.), sit  $abn >$  angulo dato. Ducatur porro  $cp \parallel am$  (§. 34.), fiantque  $nbq$ ,  $pcd$  utrumque æquale angulo dato; et  $bq$ ,  $cd$  se mutuo secabunt. Secet enim  $bq$ , (quod *per constr.* in  $nbc$  cadit) ipsam  $cp$  in  $e$ ; erit (propter  $bn \parallel cp$ )  $ebc < ecb$ , adeoque  $ec < eb$ . Sint

$$ef = ec, \text{ } efr = ecd, \text{ et } fs \parallel ep;$$

cadet  $fs$  in  $bfr$ . Nam cum  $bn \parallel cp$ , adeoque  $bn \parallel ep$ , atque  $bn \parallel fs$  sit;



erit (§. 14.)

$$\text{fbn} + \text{bfs} < 2R = \text{fbn} + \text{bfr};$$

itaque  $\text{bfs} < \text{bfr}$ . Quamobrem  $\text{fr}$  secat  $\text{ep}$ , adeoque  $\text{cd}$  quoque ipsam  $\text{eq}$  in puncto aliquo  $\delta$ .

Sit iam  $\delta g = \delta c$ , atque  $\delta g t = \delta c p = g b n$ ; erit (cum  $\text{cd} \simeq \text{gd}$  sit)

$$\text{bn} \simeq \text{gt} \simeq \text{cp}.$$

Si fuerit lineæ  $L$  formis ipsius  $\text{bn}$ , punctum in  $\text{bq}$  cadens  $\text{f}$  (§. 19.), et axis  $\text{fl}$ ; erit

$$\text{bn} \simeq \text{fl},$$

adeoque

$$\text{bfl} = \text{bgt} = \delta c p;$$

sed etiam

$$\text{fl} \simeq \text{cp}:$$

cadit ergo  $\text{f}$  manifesto in  $g$ , estque  $\text{gt} \parallel \text{bn}$ . Si vero  $\text{ho}$  ipsum  $\text{bg}$  perpendiculariter bisecet; erit  $\text{ho}$   $\text{bn}$  constructum.

### §. 36.

Fig. 10. Si fuerint data recta  $\text{cp}$  et planum  $\overline{\text{mab}}$ , atque fiat  $\text{cb} \perp \overline{\text{mab}}$ , (in  $\overline{\text{bcp}}$ )  $\text{bn} \perp \text{bc}$ , et  $\text{cq} \parallel \text{bn}$  (§. 34.); *sectio ipsius*  $\text{cp}$  (si hæc in  $\text{bcq}$  cadat) *cum*  $\text{bn}$  (in  $\overline{\text{cnn}}$ ), adeoque *cum*  $\overline{\text{mab}}$  reperitur. Et si fuerint data duo plana  $\overline{\text{pcq}}$ ,  $\overline{\text{mab}}$ , et sit  $\text{cb} \perp \overline{\text{mab}}$ ,  $\text{cr} \perp \overline{\text{pcq}}$ , atque (in  $\overline{\text{bcr}}$ )  $\text{bn} \perp \text{bc}$ ,  $\text{cs} \perp \text{cr}$ ; cadent  $\text{bn}$  in  $\overline{\text{mab}}$ , et  $\text{cs}$  in  $\overline{\text{pcq}}$ ; et sectione ipsarum  $\overline{\text{bn}}$ ,  $\overline{\text{cs}}$  (si detur) reperiata, erit perpendicularis in  $\text{pcq}$  per eandem ad  $\text{cs}$  ducta manifesto *sectio ipsorum*  $\overline{\text{mab}}$ ,  $\overline{\text{pcq}}$ .

### §. 37.

Fig. 7. In  $\overline{\text{am}}$   $\text{bn}$  reperitur tale  $a$ , ut sit  $\text{am} \simeq \text{bn}$ ; si (per §. 34.) construatur extra  $\overline{\text{nbm}}$   $\text{gt} \parallel \text{bn}$ , et fiant  $\text{bg} \perp \text{gt}$ ,  $\text{gc} = \text{gb}$ , atque  $\text{cp} \parallel \text{gt}$ ; ponaturque  $\text{tg}\delta$  ita, ut efficiat cum  $\text{tg}\text{b}$  angulum illi æqualem, quem  $\text{pc}\alpha$  cum  $\text{pc}\text{b}$  facit; atque quæretur (per §. 36.) sectio  $\overline{\text{dq}}$  ipsorum  $\text{tg}\delta$ ,  $\text{nb}\alpha$ ; fiatque

$ba \perp \delta q$ . Erit enimvero ob triangulorum  $L$  lineorum in  $F$  ipsius  $bn$  exortorum similitudinem (§. 21.) manifesto  $\delta b = \delta a$ , et  $am \triangleq bn$ .

Facile hinc patet ( $L$  lineis per *solos terminos* datis) reperiri posse etiam *terminos* proportionis quartum ac medium, atque omnes constructiones geometricas, quæ in  $\Sigma'$  in plano fiunt, hoc modo in  $F$  *absque* *XI. Axiomate* perfici posse. Ita e. g.  $4R$  in quotvis partes æquales geometricè dividi potest, si sectionem istam in  $\Sigma'$  perficere licet.

## §. 38.

Si construatur (per §. 37.) e. g.  $nbq = \frac{1}{3} R$ , et fiat (per §. 35.) in  $S$  ad  $b\bar{q}$  perpendicularis  $am \parallel bn$ , atque determinetur (per §. 37.)  $jm \triangleq bn$ ; erit, si  $ja = x$  sit, (§. 28.)

$$X = 1 : \sin. \frac{1}{3} R = 2,$$

atque  $x$  *geometricè* constructum.

Et potest  $nbq$  ita computari, ut  $ja$  ab  $i$  quovis dato minus discrepet, cum nonnisi  $\sin. nbq = \frac{1}{e}$  esse debeat.

## §. 39.

Si fuerint (in plano)  $pq$  et  $st$ ,  $\parallel$  rectæ  $mn$  (§. 27.), et  $ab$ ,  $cd$  sint perpendiculares ad  $mn$  æquales; manifesto est

$$\triangle dec \triangleq \triangle bea,$$

adeoque anguli (forsan mixtilinei)  $ecp$ ,  $eat$  congruent, atque

$$ec = ea.$$

Si porro  $cf = ag$ , erit

$$\triangle acf \triangleq \triangle cag,$$

et utrumque *quadrilateri*  $facg$  dimidium est. Si  $facg$ ,  $hagf$  duo eiusmodi quadrilatera fuerint ad  $ag$ , inter  $pq$  et  $st$ ; æqualitas eorum (uti apud EUCLIDEM), nec non triangulorum  $agc$ ,  $agh$  eidem  $ag$  insistentium,

verticesque in  $\overline{pq}$  habentium, æqualitas patet. Est porro

$$\begin{aligned} \text{atque} \quad & acf = cag, \quad gcq = cga, \\ & acf + acg + gcq = 2R \\ (\S. 32.), \text{ adeoque etiam} \quad & cag + acg + cga = 2R; \end{aligned}$$

itaque in quovis eiusmodi triangulo  $acg$  summa trium angulorum  $= 2R$ .

Sive in  $ag$  (quæ  $\parallel mn$ ) ceciderit autem *recta*  $ag$ , sive non; triangulorum *rectilineorum*  $agc$ ,  $agh$  *tam ipsorum, quam summarum angulorum ipsorumdem, æqualitas* in aperto est.

#### §. 40.

Fig. 20. *Aequalia triangula  $abc$ ,  $abd$  (abhinc rectilinea) uno latere aequali gaudentia, summas angulorum aequales habent.*

Nam dividat  $mn$  bifariam tam  $ac$  quam  $bc$ , et sit  $pq$  (per  $c$ )  $\parallel mn$ ; cadet  $d$  in  $\overline{pq}$ . Nam si  $\overline{bd}$  ipsum  $\overline{mn}$  in puncto  $e$ , adeoque (§. 39.) ipsum  $\overline{pq}$  ad distantiam  $ef = eb$  secet; erit

$$\triangle abc = \triangle abf,$$

adeoque et

$$\triangle abd = \triangle abf,$$

unde  $d$  in  $f$  cadit: si vero  $\overline{bd}$  ipsum  $\overline{mn}$  non secuerit, sit  $c$  punctum, ubi perpendicularis rectam  $ab$  bisecans ipsum  $\overline{pq}$  secat, atque  $gs = ht$  ita, ut  $\overline{st}$  *productam*  $\overline{bd}$  in puncto aliquo  $f$  secet (quod fieri posse modo simili patet, ut §. 4.); sint porro  $sl = sa$ ,  $lo \parallel st$ , atque  $o$  sectio ipsorum  $\overline{bf}$  et  $\overline{lo}$ ; esset tum (§. 39.)

$$\triangle abl = \triangle abo,$$

adeoque

$$\triangle abc > \triangle abd$$

(contra hyp.).

## §. 41.

*Aequalia triangula abc, def aequalibus angulorum summis gaudent.* Fig. 21.

Nam secet mn tam ac quam bc, ita pq tam df quam fe bifariam, et sit rs || mn, atque to || pq; erit perpendicularis ag ad rs aut æqualis perpendiculari dh ad to, aut altera e. g. dh erit maior: in quovis casu o df e centro a cum g punctum aliquod f commune habet, eritque (§. 39.)

$$\triangle abf = \triangle abc = \triangle def.$$

Est vero  $\triangle afb$  (per §. 40.) triangulo dfe, ac (per §. 39.) triangulo abc æquiangulum. Sunt igitur etiam triangula abc, def æquiangula.

In *S* converti quoque theorema potest. Sint enim triangula abc, def reciproce æquiangula, atque  $\triangle bal = \triangle def$ ; erit (per præc.) alterum alteri, adeoque etiam  $\triangle abc$  triangulo abl æquiangulum, et hinc manifesto

$$bcl + blc + cbl = 2R.$$

Atqui (ex §. 31.) cuiusvis trianguli angulorum summa in *S* est  $< 2R$ : cadit igitur l in c.

## §. 42.

*Si fuerit complementum summae angulorum trianguli abc ad 2R* Fig. 22.

*trianguli def vero*

*u,*

*v;*

*est*

$$\triangle abc : \triangle def = u : v.$$

Nam si quodvis triangulorum acg, gch, hcb, dft, ffe sit = *p*, atque

$$\triangle abc = mp, \quad \triangle def = np;$$

sitque *s* summa angulorum cuiusvis trianguli, quod = *p* est: erit manifesto

$$2R - u = ms - (m - 1)2R = 2R - m(2R - s),$$

et

$$u = m(2R - s),$$

et pariter

$$v = n(2R - s).$$

Est igitur

$$\triangle abc : \triangle def = m : n = u : v.$$

Ad casum incommensurabilitatis triangulorum  $abc$ ,  $def$  quoque extendi facile patet.

Eodem modo demonstratur triangula in superficie sphærica esse uti *excessus* summarum angulorum eorundem supra  $2R$ . Si duo anguli trianguli sphærici recti fuerint, tertius  $z$  erit excessus dictus; est autem triangulum istud (periphæria maxima  $p$  dicta) manifesto

$$= \frac{z}{2\pi} \cdot \frac{p^2}{2\pi} \quad (\S. 32. VI.);$$

consequenter quodvis triangulum, cuius angulorum excessus  $= z$ , est

$$= \frac{zp^2}{4\pi^2}.$$

#### §. 43.

Fig. 15. Iam *area* trianguli rectilinei in  $S$  per summam angulorum exprimetur.

Si  $ab$  crescat in infinitum; erit (§. 42.)

$$\triangle abc : (R - u - v)$$

constans. Est vero

$$\triangle abc \sim bacn \quad (\S. 32. V.)$$

et

$$R - u - v \sim z \quad (\S. I.);$$

adeoque

$$bacn : z = \triangle abc : (R - u - v) = bac'n' : z'.$$

Est porro manifesto

$$bdcn : bd'c'n' = r : r' = \text{tang. } z : \text{tang. } z' \quad (\S. 30.).$$

Pro  $y' \sim 0$  autem est

$$\frac{bd'c'n'}{bac'n'} \sim 1,$$

nec non

$$\frac{\text{tang. } z'}{z'} \sim 1;$$

consequ.

$$bdcn : bacn = \text{tang. } z : z.$$

Erat vero (§. 32.)

$$bdcn = ri = i^2 \text{ tang. } z;$$

est igitur

$$bacn = zi^2.$$

Quovis triangulo, cuius angulorum summæ complementum ad  $2R$   $z$  est, in posterum breviter  $\Delta$  dicto, erit idcirco

$$\Delta = zi^2.$$

Facile hinc liquet, quod si

Fig. 14.

$$\text{or} \parallel \text{am} \quad \text{et} \quad \text{ro} \parallel \text{ab}$$

fuerint; *area* inter  $\overline{or}$ ,  $\overline{st}$ ,  $\overline{bc}$  comprehensa (quæ manifesto limes absolutus est *area* triangulorum rectilineorum sine fine crescentium, seu ipsius  $\Delta$  pro  $z \sim 2R$ ), sit

$$= \pi i^2 = \odot i \text{ in } F.$$

Limite isto per  $\square$  denotato, erit porro (per §. 30.)

Fig. 15.

$$\pi r^2 = \text{tang. } z^2 \square = \odot r \text{ in } F \text{ (§. 21.)}$$

$$= \odot s \text{ (per §. 32. VI.),}$$

si chorda  $dc$   $s$  dicatur. Si iam radio dato  $s$ , circuli in plano (sive radio  $L$  formi circuli in  $F$ ) perpendiculariter bisecto, construat (per §. 34.)  $db \parallel \simeq cn$ ; demissa perpendiculari  $ca$  ad  $db$ , et erecta perpendiculari  $cm$  ad  $ca$ ; habebitur  $z$ ; unde (per §. 37.)  $\text{tang. } z^2$ , radio  $L$  formi ad lubitum pro unitate assumto, *geometrice determinari potest per duas lineas uniformes eiusdem curvaturae* (quæ solis terminis datis, constructis axibus, manifesto tanquam rectæ commensurari, atque hoc respectu rectis æquivalentes spectari possunt).

Fig. 23. Porro construitur quadrilaterum ex. gr. regulare  $=\square$ , ut sequitur. Sit

$$abc = R, \quad bac = \frac{1}{2} R, \quad acb = \frac{1}{4} R, \quad \text{et } bc = x;$$

poterit  $X$  (ex §. 31. II.) per meras radices quadraticas exprimi, et (per §. 37.) construi: habitoque  $X$ , (per §. 38., sive etiam 29. et 35.)  $x$  ipsum determinari potest. Estque octuplum  $\triangle abc$  manifesto  $=\square$ , atque *per hoc, circulus planus radii  $s$ , per figuram rectilineam, et lineas uniformes eiusdem generis (rectis, quoad comparisonem inter se, æquivalentes) geometricè quadratus; circulus  $F$ formis vero eodem modo complanatus: habeturque aut Axioma XI. Euclidis verum, aut quadratura circuli geometrica*; etsi hucusque indecisum manserit, quodnam ex his duobus revera locum habeat. Quoties  $\text{tang. } z^2$  *vel* numerus integer *vel* fractio rationalis fuerit, cuius (ad simplicissimam formam reductæ) denominator *aut* numerus primus formæ  $2^m + 1$  (cuius est etiam  $2 = 2^0 + 1$ ) *aut* productum fuerit e quotcunque primis huius formæ, quorum (ipsum 2, qui solus quotvis vicibus occurrere potest, excipiendo) quivis *semel* ut factor occurrit: per theoriam polygonorum ill. *GAUSS* (præclarum nostri imo omnis ævi inventum), etiam ipsi  $\text{tang. } z^2 \square = \odot s$  (et nonnisi pro talibus valoribus ipsius  $z$ ) figuram rectilineam æqualem constituere licet. Nam *divisio* ipsius  $\square$  (theoremate §. 42. facile ad quælibet polygona extenso) manifesto *sectionem* ipsius  $2R$  requirit, quam (ut ostendi potest) unice sub dicta conditione geometricè perficere licet. In omnibus autem talibus casibus præcedentia facile ad scopum perducent. Et potest quævis figura rectilinea in polygonum regulare  $n$  laterum geometricè converti, siquidem  $n$  sub formam *GAUSSianam* cadat.

Superesset denique, (ut res omni numero absolvatur), impossibilitatem (absque suppositione aliqua) decidendi, num  $\Sigma'$  aut aliquod (et quodnam)  $S$  sit, demonstrare: quod tamen occasione magis idoneæ reservatur.

WOLFGANGI BOLYAI  
ADDITAMENTUM AD APPENDICEM.

Denique *aliquid Auctori Appendicis proprium, coronidis instar*, addere fas sit: qui tamen ignoscat, si quid non acu eius tetigerim.

Res breviter in eo consistit: *formulae trigonometriae sphaericae*, in Appendice dicta ab axioma XI. Eucl. independenter demonstratæ, *cum formulis trigonometriae planae conveniunt, si* (modo statim dicendo) *latera trianguli sphaerici realia, rectilinei vero imaginaria accipiantur*; adeo ut quoad formulas trigonometricas planum ut sphaera imaginaria considerari possit, si pro reali illa accipiat, in qua  $\sin. R=1$ .

Pro casu, si axioma Eucl. verum non fuerit, demonstratur (Appendix §. 30.) dari certum  $i$ , pro quo ibidem dictum  $I$  est  $=e$  (basi logarithmorum naturalium), atque pro hoc casu formulæ trigonometriæ planæ quoque demonstrantur (ibidem §. 31.); et quidem ita, ut (iuxta §. 32., post VII., ibidem) formulæ et pro casu veritatis axiomatis dicti valeant; nempe si supponendo, quod  $i \rightarrow \infty$ , limites valorum accipiantur; nimirum systema Euclideum est quasi limes systematis antieuclidei (pro  $i \rightarrow \infty$ ). Ponatur, pro casu existentis  $i$ , unitas  $=i$ , atque conceptus *sinus cosinusque* extendatur et ad arcus imaginarios; ita ut arcum sive realem sive imaginarium denotet  $p$ , dicatur

$$\frac{1}{2}(e^{p\sqrt{-1}} + e^{-p\sqrt{-1}})$$

cosinus ipsius  $p$ , et

$$\frac{1}{2\sqrt{-1}}(e^{p\sqrt{-1}} - e^{-p\sqrt{-1}})$$

dicatur sinus ipsius  $p$ .



Erit hinc pro  $q$  reali

$$\begin{aligned}\frac{1}{2\sqrt{-1}}(e^q - e^{-q}) &= \frac{1}{2\sqrt{-1}}(e^{-q\sqrt{-1}\sqrt{-1}} - e^{q\sqrt{-1}\sqrt{-1}}) = \\ &= \sin.(-q\sqrt{-1}) = -\sin. q\sqrt{-1}\end{aligned}$$

Ita

$$\begin{aligned}\frac{1}{2}(e^q + e^{-q}) &= \frac{1}{2}(e^{-q\sqrt{-1}\sqrt{-1}} + e^{q\sqrt{-1}\sqrt{-1}}) = \\ &= \cos.(-q\sqrt{-1}) = \cos. q\sqrt{-1};\end{aligned}$$

si nempe et in circulo imaginario sinus negativi arcus sinui arcus positivi alioquin priori æqualis sit, præterquam quod negativus sit, atque cosinus arcus positivi et negativi (si alioquin æquales fuerint), sit idem.

In Appendice dicta §. 25. demonstratur absolute, id est ab axioma dicto independenter; quod in quovis triangulo rectilineo *sinus angulorum sint, uti peripheriae radiorum lateribus oppositis æqualium*; demonstraturque porro, pro casu existentis  $i$ , peripheriam radii  $y$  esse

$$= \pi i (e^{\frac{y}{i}} - e^{-\frac{y}{i}}),$$

quod pro  $i=1$  fit

$$\pi(e^y - e^{-y}).$$

Itaque (§. 31. ibidem) pro triangulo rectilineo rectangulo, cuius catheti sunt  $a$  et  $b$ , hypotenusa  $c$ , et anguli lateribus  $a$ ,  $b$ ,  $c$  oppositi sunt  $\alpha$ ,  $\beta$ ,  $\pi$ ; est (pro  $i=1$ )

in I.

$$1 : \sin. \alpha = \pi(e^c - e^{-c}) : \pi(e^a - e^{-a});$$

adeoque

$$1 : \sin. \alpha = \frac{1}{2\sqrt{-1}}(e^c - e^{-c}) : \frac{1}{2\sqrt{-1}}(e^a - e^{-a}).$$

Unde

$$1 : \sin. \alpha = -\sin. c\sqrt{-1} : -\sin. a\sqrt{-1}.$$

Et hinc

$$1 : \sin. \alpha = \sin. c\sqrt{-1} : \sin. a\sqrt{-1}.$$

In II. fit

$$\cos. \alpha : \sin. \beta = \cos. a \sqrt{-1} : 1.$$

In III. fit

$$\cos. c \sqrt{-1} = \cos. a \sqrt{-1} \cos. b \sqrt{-1}.$$

Quæ prouti omnes exinde promanantes formulæ trigonometriæ planæ, cum formulis trigonometriæ sphæricæ prorsus conveniunt; nisi quod si ex. gr. trianguli sphærici rectanguli quoque catheti angulique iis oppositi, hypotenusæque nomina eadem sortiantur, latera trianguli rectilinei per  $\sqrt{-1}$  dividenda sint, ut formulæ pro sphæricis prodeant.

Nempe ex I. fiet

$$1 : \sin. \alpha = \sin. c : \sin. a,$$

ex II. fiet

$$1 : \cos. a = \sin. \beta : \cos. \alpha,$$

ex III. fiet

$$\cos. c = \cos. a \cos. b.$$

Quum ceteris supersedere liceat, et lectorem deductione (App. §. 32. post VII.) ommissa offendi impediri que expertus sim: haud abs re erit ostendere, quomodo ex. gr. ex

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2} (e^{\frac{a}{i}} + e^{-\frac{a}{i}}) (e^{\frac{b}{i}} + e^{-\frac{b}{i}})$$

sequatur

$$c^2 = a^2 + b^2$$

(theorema Pythagoreum pro systemate Euclideo); verosimiliter Auctor quoque ita deduxit, et ceteræ quoque eodem modo sequuntur.

Est nempe potentiis ipsius  $e$  per series expressis

$$e^{\frac{k}{i}} = 1 + \frac{k}{i} + \frac{k^2}{2i^2} + \frac{k^3}{2 \cdot 3i^3} + \frac{k^4}{2 \cdot 3 \cdot 4i^4} + \dots$$

$$e^{-\frac{k}{i}} = 1 - \frac{k}{i} + \frac{k^2}{2i^2} - \frac{k^3}{2 \cdot 3i^3} + \frac{k^4}{2 \cdot 3 \cdot 4i^4} - \dots;$$

adeoque

$$e^{\frac{k}{i}} + e^{-\frac{k}{i}} = 2 + \frac{k^2}{i^2} + \frac{k^4}{3 \cdot 4i^4} + \dots$$

$$= 2 + \frac{k^2 + u}{i^2},$$

(si omnium terminorum post  $\frac{k^2}{i^2}$  summa  $\frac{u}{i^2}$  dicatur); estque  $u \rightarrow 0$ , dum  $i \rightarrow \infty$ . Nam multiplicentur omnes termini post  $\frac{k^2}{i^2}$  per  $i^2$ ; erit terminus primus  $\frac{k^4}{3 \cdot 4 i^2}$ , et quivis exponens  $< \frac{k^2}{i^2}$ ; essetque etsi exponens ubique hic maneret, summa

$$\frac{k^4}{3 \cdot 4 i^2} : \left(1 - \frac{k^2}{i^2}\right) = \frac{k^4}{3 \cdot 4 (i^2 - k^2)},$$

quod manifesto  $\rightarrow 0$ , dum  $i \rightarrow \infty$ .

Atque ex

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2} \left( e^{\frac{a+b}{i}} + e^{-\frac{a+b}{i}} + e^{\frac{a-b}{i}} + e^{-\frac{a-b}{i}} \right)$$

sequitur (pro  $\omega$ ,  $v$ ,  $\lambda$  adinstar  $u$  acceptis)

$$2 + \frac{c^2 + \omega}{i^2} = 1 + \frac{(a+b)^2 + v}{2i^2} + 1 + \frac{(a-b)^2 + \lambda}{2i^2}.$$

Atque hinc

$$c^2 = \frac{1}{2} (a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + v + \lambda - 2\omega),$$

quod

$$\rightarrow a^2 + b^2.$$

*Scholion.* Sphæræ illius, in qua sinus totus est  $1=i$ , radius est ordinata  $y$  lineæ  $L$  formis ipsi  $i=1$  æqualis, ad axem per unam extremitatem ex altera perpendiculariter missa. Nempe *in superficie* (App. §. 21.) *F dicta, tota Geometria Euclidea valet, lineis L vicem rectarum subeuntibus*: atque pro radio  $L$  formi  $=1$ , qui sinus totus in  $F$  erit, peripheriæ eiusdem radius in plano erit plane dictum  $y$ ; quod ad sphæram imaginariam, ad quam planum (in systemate antieuclydeo) revocatur, facile applicatur.

## ADNOTATIONES EDITORUM.

In Ed. I. Appendicis literæ singulares puncta denotantes literis quæ dicuntur *cursivis* impressæ sunt, sed in libellis nobis relictis manu Ioannis Bolyai scriptis puncta literis qu. d. *fractur* denotantur, quibus etiam pater eius usus est.

*Pag. 1. post §. 15. in Ed. I. legitur :*

$\perp$  denotet perpendiculare  
 $\wedge$  « angulum.

Nos hæc delevimus, quia pro signo  $\perp$  usitatio  $\bot$  utimur, vocabulum «angulum» autem ubique integrum scribimus.

*Pag. 8. §. ultimo «frs»* correximus in «hrs», ut et ipse auctor correxerat in libello manu scripto, quo Appendicem lingua Germanica denuo pertractavit.

*Pag. 10. §. ultimo §. 18. «○ha»* emendavimus ex «○hf».

*Pag. 14. extremæ §. 26.* auctor ascripsit exemplari in bibliotheca Academicæ Hungaricæ asservato «cos. quoque necessarium». Videtur in editione quadam altera propositionem de cosinu demonstraturus fuisse, quamquam nihil hic maioris momenti vere desideratur. Revera considerationes, quibus in «Supplemento numeri 31351» Tom. II. Tentaminis ex

$$\sin. H : \sin. A = 1 : \sin. a$$

trigonometria sphærica integra deducitur, et in geometria absoluta valent.

*Pag. 16. §. 3.* Ex libello Germanico auctoris manu scripto «cq» scripsimus pro vitioso «cg» Editionis I.

*Pag. 17. §. 5.* «cn || ab, c'n' || ab» scripsimus pro «cn, c'n' || ab» Ed. I.

*Pag. 20. §. 8.* «et 27» Ioannes Bolyai ipse ascripsit in exemplari Academiæ.

*Pag. 22. §. 4. a calce* «§. 30.» scripsimus pro «§. 29.» Ed. I.

*Pag. 35.* In hoc Additamento, quod pagg. 380—383 Tom. II. Ed. I. Tentaminis continetur, delevimus mentionem de paginis huius operis factam, in quibus solum theoremata omnibus nota continentur. Articulum vero secundum in hunc locum reiecimus:

«Pro  $v$  positivo radicem positivam ipsius  $-v^2$  per  $+v$  denotat, et negativam per  $-v$ , sed ob defectum signorum (quum vix hæc duo quadamtenus prodierunt), radix positiva ipsius  $-1$  per  $\sqrt{-1}$  et negativa per  $-\sqrt{-1}$  denotabitur».

*Ibidem §. 3.* Post «Appendicis» delevimus «in tomo primo».

*Ibidem §. 11.* Initium articuli huius in Ed. I. sic legitur:

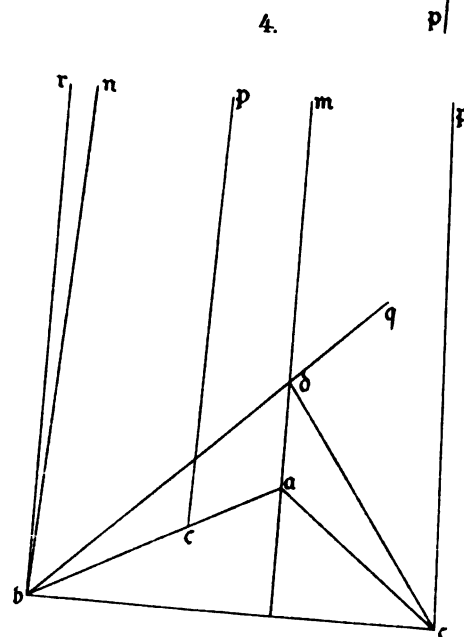
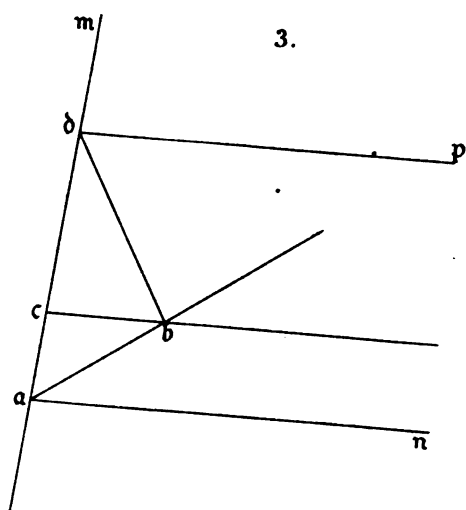
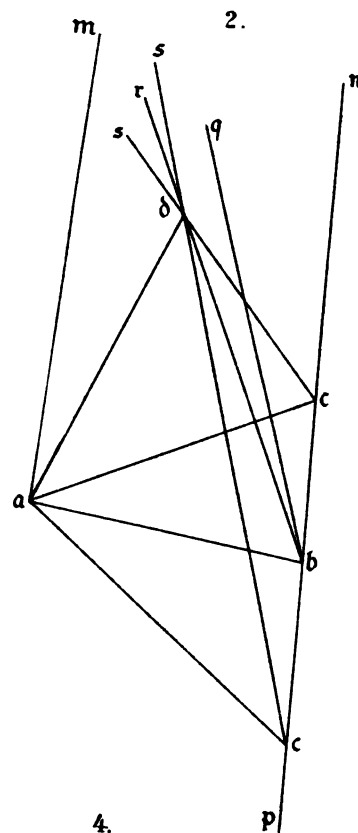
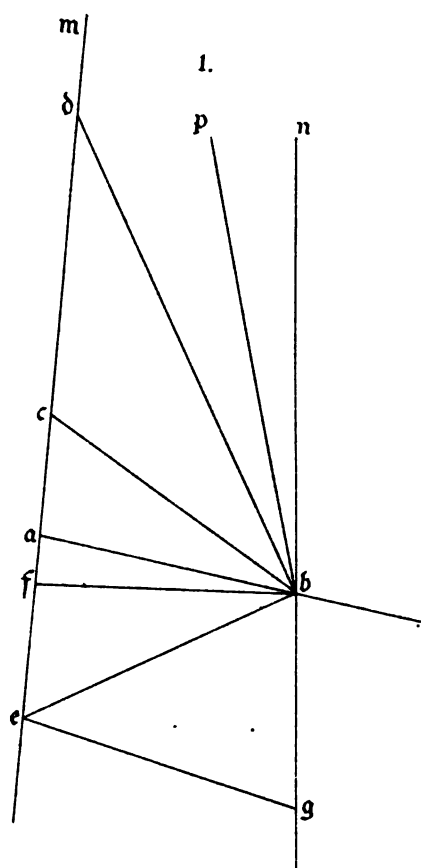
«Nimirum de aximate Euclideo dictum in tomo primo satis superque est: pro casu, si verum non fuerit ☞».

*Pag. 38. §. 2.* «multiplicentur» scripsimus pro erroneo «dividantur».

*Ibidem §. 8.* Latus sinistrum formulæ in Ed. I. deest.

I. BOLYAI, Appendix.

Tab. I.

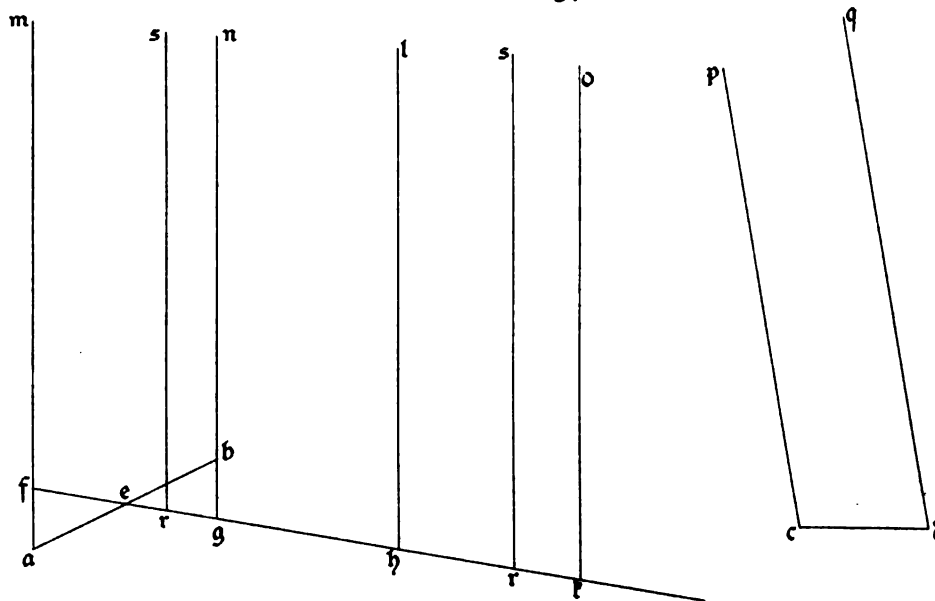


Del. TÖTÖSSY B.

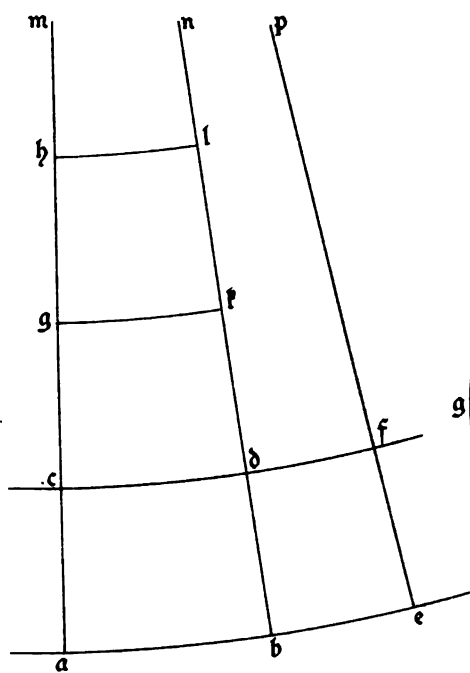
Lith. GRUND V. utódai.

**Tab. II. Fig. 5-7.**

8.

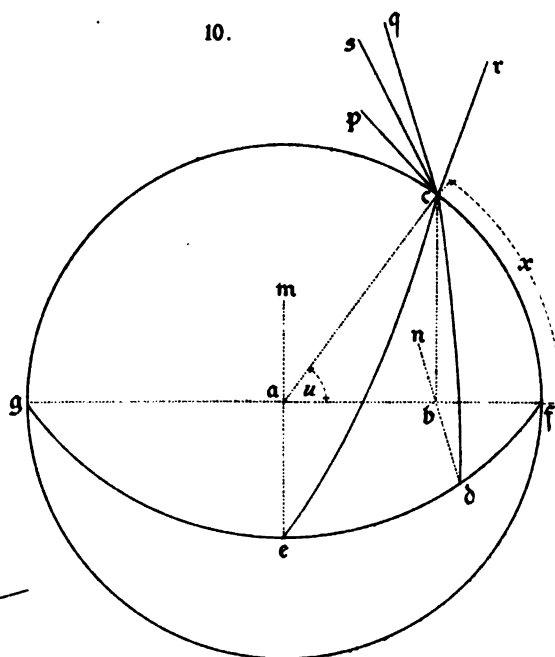


9.



Del. TÖTÖSSY B.

10.

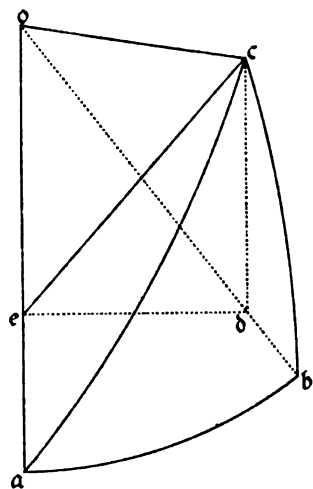


Lith. GRUND V. utódaí.

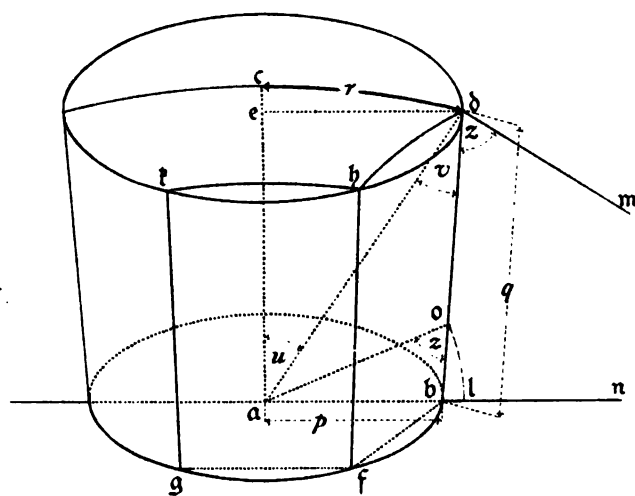


Tab. III. Fig. 8-10. . .

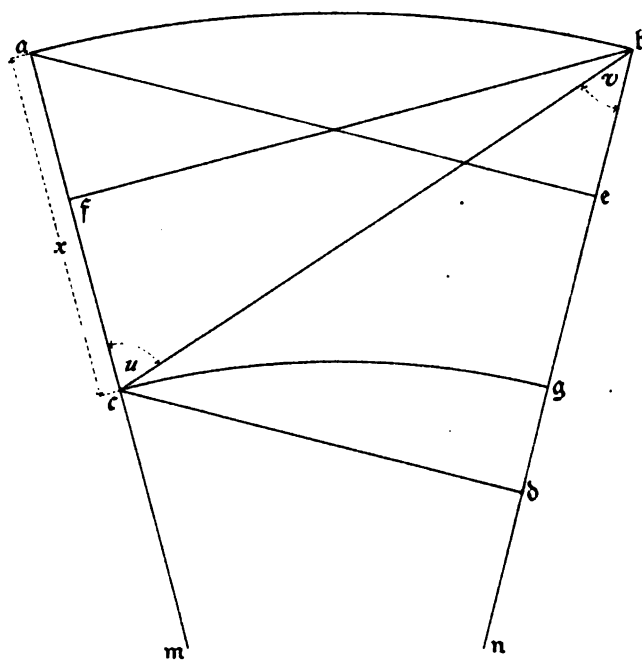
**11.**



**12.**

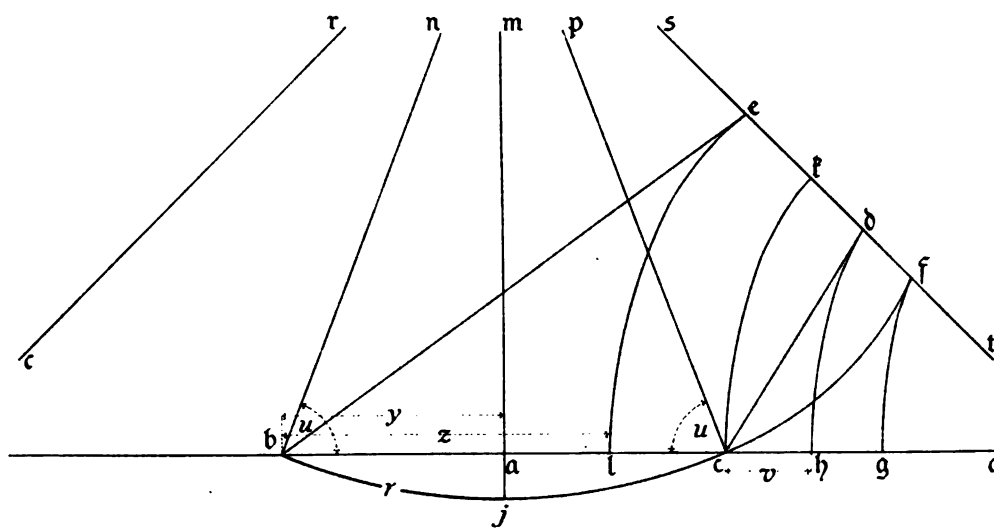


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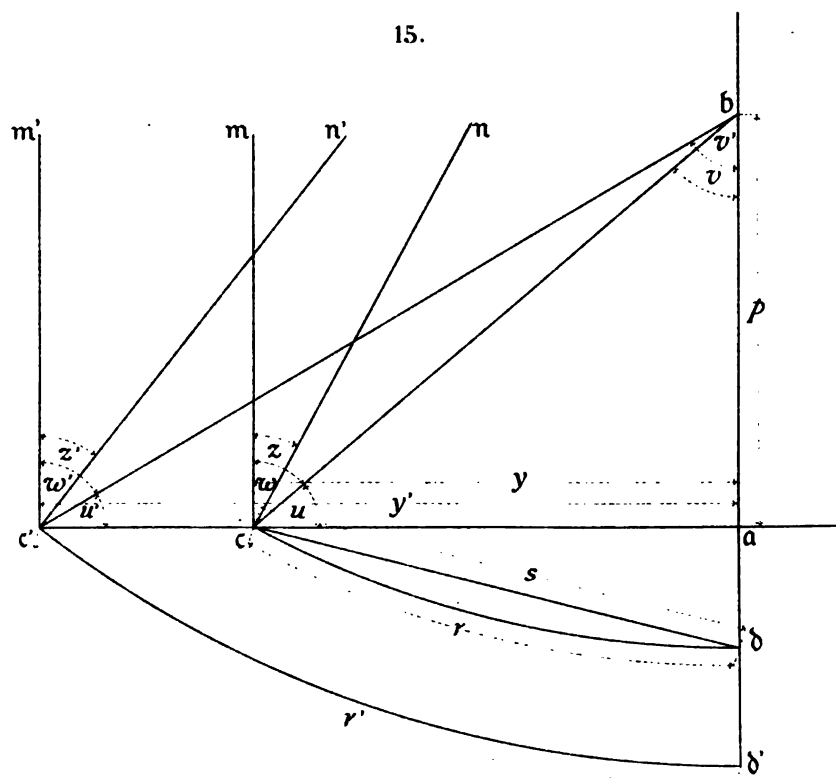


Tab. IV. Fig. 11-13.

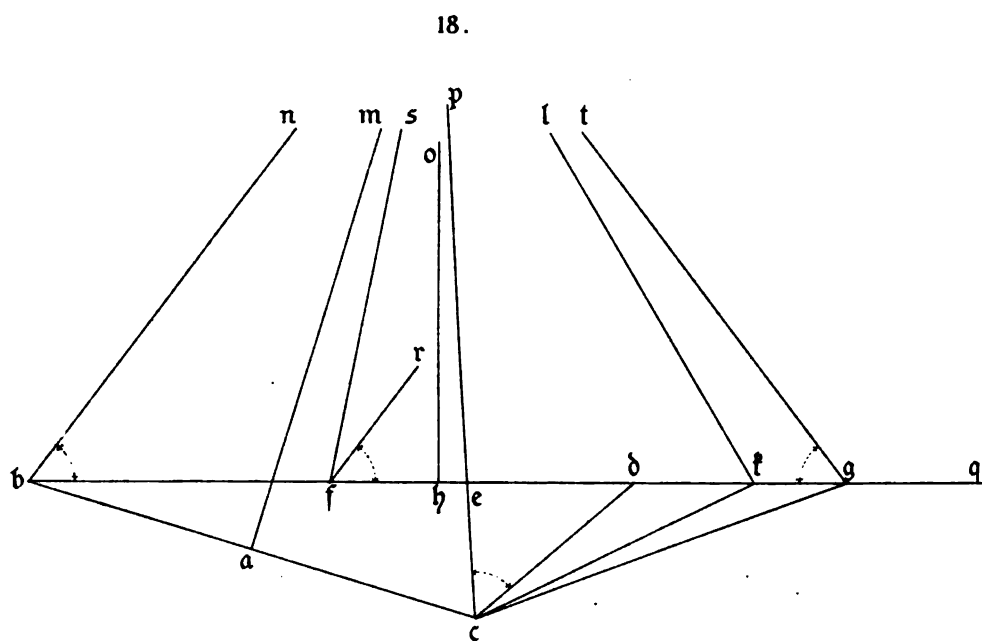
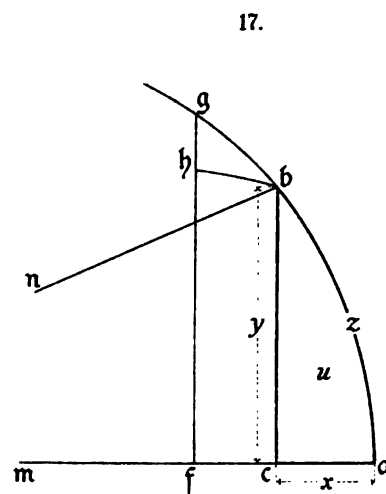
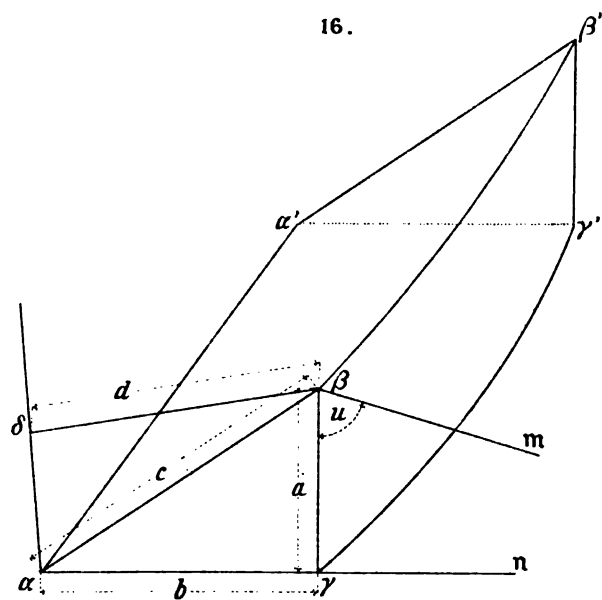
14.



15.

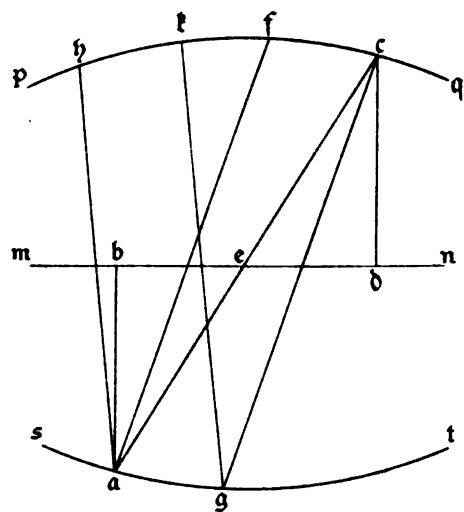


Tab. V. Fig. 14-15.

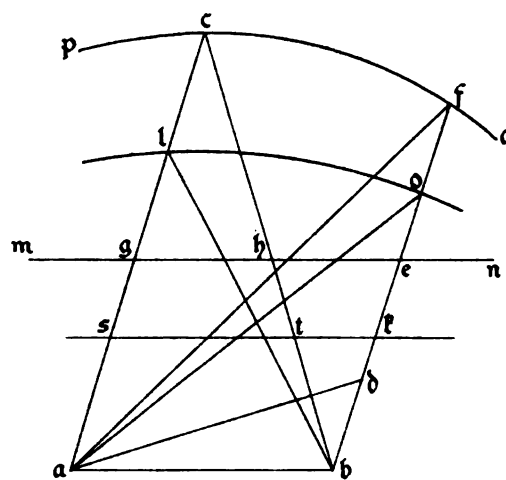


Tab. VI. Fig. 16.-18.

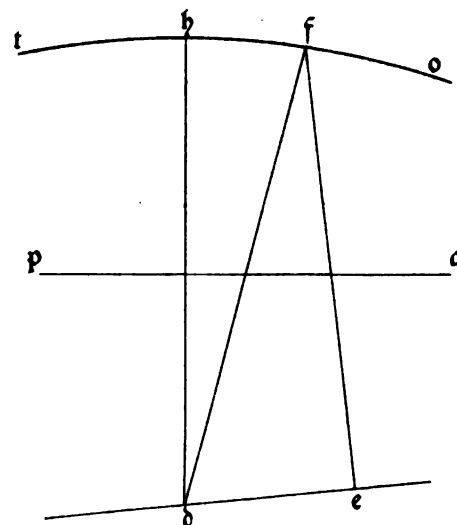
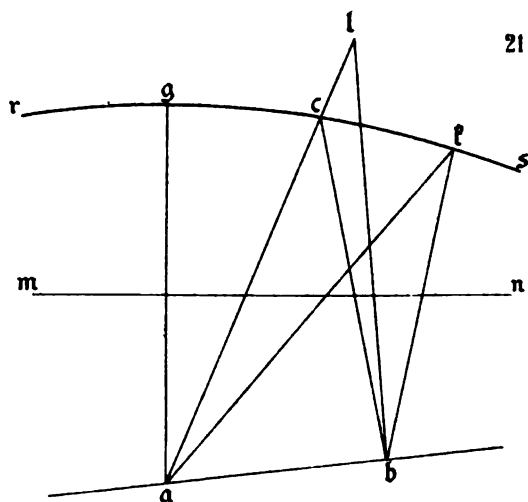
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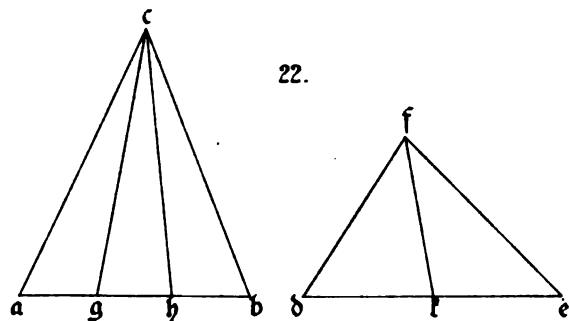
20.



21.

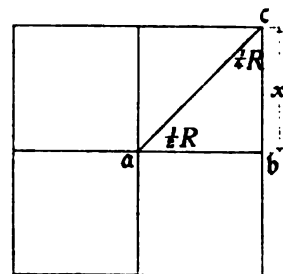


22.



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Tab. VII. Fig. 19.-23.





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